MASKIN MONOTONIC AGGREGATION RULES

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14 February 2005

rev. 20 September 2005

ABSTRACT[1]

Given a society confronting two alternatives, we show that the set of anonymous, neutral and Maskin monotonic aggregation rules coincides with the family of absolute qualified majority rules. We also explore the effect of incorporating Pareto optimality in our characterization.

JEL Classification: D71

Key Words: Majority rule, Maskin monotonicity

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[1] We thank Eric Maskin and two anonymous referees whose comments led to an improvement of our paper. Naturally, the authors are responsible from all possibly remaining errors.

1. INTRODUCTION

Given a society confronting two alternatives x and y, it is possible to attribute two essentially incompatible meanings to the concept of a majoritarian decision. One of these is based on the idea of a relative majority so that in deciding on the social ranking of x and y, it is the relative number of x and y supporters which matters. A second concept of majoritarianism is based on the idea of an absolute majority where the social ranking between x and y depends on some absolute number that the supporters of these alternatives must reach.

Relative majority rules have been first characterized by May (1952) and later by Aşan and Sanver (2002) and Woeginger (2003). A well-known characterization of absolute majority rules is due to Austen-Smith and Banks (1999) while Yi (2005) gives an alternative characterization of absolute majoritarianism.[2] Our paper is an exploration in the field of absolute majority rules. We aim to generalize and to some extent clarify the results of Yi (2005) who characterizes "a (weak) majority rule" in terms of anonymity, neutrality, independence of irrelevant alternatives and "positive responsiveness". While the first three of these conditions carry their standard meanings, what Yi (2005) calls "positive responsiveness" is equivalent to the well-known monotonicity condition of Maskin (1999).[3] So we consider a world of two alternatives to show that Maskin monotonicity, combined with anonymity and neutrality, characterizes the family of absolute (qualified) majority rules.[4] This result, which is a characterization of Maskin monotonic aggregation rules, also points to the fact that what Yi (2005) calls "a weak majority rule" turns out to coincide with the concept of an absolute majority rule.

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[2] When indifferences are ruled and an odd number of agents is assumed, absolute and relative majoritarianism coincide. For other characterizations of majority rules in various restricted frameworks where the two concepts converge to each other, one can see Maskin (1995), Dasgupta and Maskin (1998), Campbell and Kelly (2000).

[3] As much as we understand from the formal definition of the concept, Yi (2005) erroneously claims - in the last sentence of the introduction- that his positive responsiveness property is weaker than the one used by May (1952). In fact, if this were to be true, then the (relative) majority rule à la May (1952) would constitute a counter-example to his Theorem 1.

[4] Our characterization differs from the one made by Austen-Smith and Banks (1999) who conjoin anonymity and neutrality with a "decisiveness" and "monotonicity" condition, both of which differ from Maskin monotonicity.

Section 2 gives the preliminaries. Section 3 states our results. Section 4 makes some closing remarks.

2. PRELIMINARIES

Taking any natural number $n \geq 2$, we consider a society $\mathbf{N} = \{1,\ldots, n\}$ confronting a set of alternatives $\mathbf{A} = \{a, b\}$. Every $i \in \mathbf{N}$ has a complete and transitive preference $R_i \in \{-1, 0, 1\}$ over **A**.[5] A *preference profile* (of the society) is an n-tuple $R = (R_1,\ldots R_n) \in \{-1, 0, 1\}^n$ of individual preferences. An *aggregation rule* is a function $F: \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ that maps preference profiles into complete and transitive social preferences.[6]

We write $n_+(R) = \#\{i \in \mathbf{N} \mid R_i = 1\}$ for the number of agents who prefer a to b at $R \in \{-1, 0, 1\}^n$. Similarly $n_-(R) = \#\{i \in \mathbf{N} \mid R_i = -1\}$. Let n^* be the lowest integer exceeding n/2. Picking some $q \in \{n^*,\ldots, n+1\}$, we define an *absolute q-majority rule* as an aggregation rule F such that for every $R \in \{-1, 0, 1\}^n$ we have $(F(R) = 1 \Leftrightarrow n_+(R) \geq q)$ and $(F(R) = -1 \Leftrightarrow n_-(R) \geq q)$.[7] Remark that the (relative) majority rule à la May (1952) which is defined as $F(R) = \text{sgn}(\Sigma_{i \in \mathbf{N}} R_i)$ for all $R \in \{-1, 0, 1\}^n$ is not an absolute q-majority rule.[8]

3. RESULTS

The class of absolute q-majority rules can be characterized in terms of the following three (logically independent) axioms imposed over aggregation rules:

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[5] We write $R_i = 1$ (resp. $R_i = -1$) whenever agent i prefers a to b (resp. b to a) while $R_i = 0$ means that i is indifferent between a and b.

[6] So given any $R \in \{-1, 0, 1\}^n$, we interpret $F(R) = 1$ (resp. $F(R) = -1$) as "a being socially preferred to b" (resp. "b being socially preferred to a") while $F(R) = 0$ means "social indifference".

[7] As F(R) is complete, $(F(R) = 0 \Leftrightarrow n_+(R) < q$ and $n_-(R) < q)$ naturally follows. Note that taking $q = n+1$ gives a degenerate majority rule where $F(R) = 0$ at every $R \in \{-1, 0, 1\}^n$. We address the efficiency issue at the end of the section.

[8] For any real number r, sgn(r) equals 1, 0, -1 when $r > 0$, $r = 0$, $r < 0$, respectively.

Anonymity (AN): Given any $R \in \{-1, 0, 1\}^n$ and any permutation function Π: $\mathbf{N} \to \mathbf{N}$, we have $F(R_1,\dots R_n) = F(R_{\Pi(1)},\dots R_{\Pi(n)})$.

Neutrality (NE): $F(-R) = -F(R)$ for all $R \in \{-1, 0, 1\}^n$.

Maskin Monotonicity (MM): For any R, R' $\in \{-1, 0, 1\}^n$ such that $R_i \geq 0 \Rightarrow R_i' \geq 0$ for each $i \in \mathbf{N}$, we have $F(R) \geq 0 \Rightarrow F(R') \geq 0$. Similarly, for any R, R' $\in \{-1, 0, 1\}^n$ such that $R_i \leq 0 \Rightarrow R_i' \leq 0$ for each $i \in \mathbf{N}$, we have $F(R) \leq 0 \Rightarrow F(R') \leq 0$.

Note that AN and NE have their standard meanings of equally treating individuals and alternatives, respectively. Similarly, MM is the well-known monotonicity condition of Maskin (1999) which is necessary for Nash implementability.

Theorem 3.1: An aggregation rule F satisfies AN, NE and MM if and only if F is a q-majority rule for some $q \in \{n^*,\dots, n+1\}$.

Proof: The "if" part is obvious and left to the reader. To prove the "only if" part, we take any F which satisfies AN, NE and MM. To show that F is a q-majority rule, we assert the following two implications:

(i) Take any $R \in \{-1, 0, 1\}^n$. If $F(R) = 1$ then $n_+(R) \geq q$ for some $q \in \{n^*,\dots, n+1\}$. Similarly, if $F(R) = -1$ then $n_-(R) \geq q$ for some $q \in \{n^*,\dots, n+1\}$.

(ii) Take any R, R' $\in \{-1, 0, 1\}^n$. If $F(R) = 1$ and $n_+(R') \geq n_+(R)$ then $F(R') = 1$. If $F(R) = -1$ and $n_-(R') \geq n_-(R)$ then $F(R') = -1$.

The conjunction of (i) and (ii) implies the existence of $q_1, q_2 \in \{n^*,\dots, n+1\}$ such that at each $R \in \{-1, 0, 1\}^n$ we have ($F(R) = 1 \Leftrightarrow n_+(R) \geq q_1$) and ($F(R) = -1 \Leftrightarrow n_-(R) \geq q_2$). By AN and NE, q_1 and q_2 must be equal, which establishes that F is an absolute q-majority rule. So we complete the proof by showing (i) and (ii).

To see (i), take any $R \in \{-1, 0, 1\}^n$ with $F(R) = 1$ and suppose $n_+(R) < n^*$. Now consider $R' \in \{-1, 0, 1\}^n$ where for all $i \in \mathbf{N}$ we have ($R_i' = 1 \Leftrightarrow R_i = 1$) and ($R_i' = 0 \Leftrightarrow R_i \in \{-1, 0\}$). By MM, $F(R') = 1$ as well. Now take a set of voters K with cardinality $n_+(R)$ such that $i \in K \Rightarrow R_i \in \{-1, 0\}$. Consider $R'' \in \{-1, 0, 1\}^n$ where for all i $\in$ **N** we have (R_i'' = 1 $\Leftrightarrow$ R_i = 1) and (R_i'' = - 1 $\Leftrightarrow$ i $\in$ K). By AN and NE, we have F(R'') = 0 which, combined with F(R) = 1, contradicts MM. The case of F(R) = -1 can be similarly shown.

To see (ii), take any R, R' $\in$ {-1, 0, 1}^{n} with F(R) = 1 and n_+(R') ≥ n_+(R). Suppose F(R') $\in$ {-1, 0}. Now, pick the profile Q $\in$ {-1, 0, 1}^{n} where for all i $\in$ **N** we have (Q_i = 1 $\Leftrightarrow$ R_i = 1) and (Q_i = 0 $\Leftrightarrow$ R_i $\in$ {-1, 0}). By MM, we have F(Q) = 1. Similarly, pick the profile Q' $\in$ {-1, 0, 1}^{n} where for all i $\in$ **N** we have (Q_i' = 1 $\Leftrightarrow$ R_i' = 1) and (Q_i' = 0 $\Leftrightarrow$ R_i' $\in$ {-1, 0}). Again by MM, we have F(Q') $\in$ {-1, 0}. Now, take some Q'' $\in$ {-1, 0, 1}^{n} with n_+(Q'') = n_+(Q') and n_+(Q'') = 0 such that Q_i = 1 $\Rightarrow$ Q_i'' = 1 for every i $\in$ **N**. By AN, we have F(Q'') = F(Q'). Thus, F(Q'') $\in$ {-1, 0} which, combined with F(Q) = 1, contradicts MM. The case of F(R) = -1 and n_-(R') ≥ n_-(R) can be similarly shown. ■

We now explore the effects of incorporating Pareto optimality into the picture. An aggregation rule is weakly Pareto optimal (WPO) iff given any R $\in$ {-1, 0, 1}^{n}, we have F(R) = 1 whenever R_i = 1 for all i $\in$ **N** and we have F(R) = -1 whenever R_i = -1 for all i $\in$ **N**. It is clear that the only absolute q-majority rule which violates WPO is the one where q = n+1, leading to the following theorem:

Theorem 3.2: An aggregation rule F satisfies AN, NE, MM and WPO if and only if F is a q-majority rule for some q $\in$ {n*,…, n}.

Remark 3.1: It may be of interest to compare q-majority rules according to their capability of inducing ties. Given any two aggregation rules F and G, we say that *F induces less ties than G* if and only if G(R) ≠ 0 $\Rightarrow$ F(R) ≠ 0 for all R $\in$ {-1, 0, 1}^{n} while there exists Q $\in$ {-1, 0, 1}^{n} where G(Q) = 0 and F(Q) ≠ 0. It is clear that given two q-majority rules, the one with a lower q induces less ties – hence we confront the lowest number of ties when q = n*.

It is possible to define the stronger version of Pareto optimality and call an aggregation rule Pareto optimal (PO) iff given any $R \in \{-1, 0, 1\}^n$, we have ($R_i \geq 0$ for all $i \in \mathbf{N}$ while $R_j = 1$ for some $j \in \mathbf{N} \Rightarrow F(R) = 1$) and ($R_i \leq 0$ for all $i \in \mathbf{N}$ while $R_j = -1$ for some $j \in \mathbf{N} \Rightarrow F(R) = -1$). However, Maskin monotonicity and PO are incompatible, as stated in the following theorem:

Theorem 3.3: There exists no aggregation rule which satisfies MM and PO.

Proof: Suppose F is an aggregation rule which satisfies MM and PO. Take some $R \in \{-1, 0, 1\}^n$ such that $R_i = 1$ for some $i \in \mathbf{N}$ and $R_j = -1$ for some $j \in \mathbf{N}$. Consider the case where $F(R) \in \{1, 0\}$. Let $R' \in \{-1, 0, 1\}^n$ be such that for all $i \in \mathbf{N}$ we have ($R_i' = -1 \Leftrightarrow R_i = -1$) and ($R_i' = 0 \Leftrightarrow R_i \in \{1, 0\}$). PO implies $F(R') = -1$ which, combined with $F(R) \in \{1, 0\}$ contradicts MM. The case of $F(R) \in \{-1, 0\}$ leads to a similar contradiction. ■

4. CLOSING REMARKS

It is possible to extend our results to a framework with more than two alternatives where all complete preferences are allowed as social orderings. Using the standard definitions of anonymity, neutrality, Maskin monotonicity and independence of irrelevant alternatives (IIA)[9], one can show that the class of anonymous, neutral, Maskin monotonic and IIA aggregation rules coincides with those where a given absolute qualified majority rule is used to generate the social decision about each pair of alternatives – a result which generalizes Yi (2005).[10]

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[9] By anonymity, we mean that the aggregation rule is invariant to renamings of individuals, so that voters are equally treated. By neutrality, we mean that the aggregation rule reflects renamings of alternatives, so that alternatives are equally treated. Maskin monotonicity is what Yi (2005) calls "positive responsiveness" and IIA has its standard meaning proposed by Arrow (1951).

[10] To see this, first note that by IIA, the decision over each pair of alternatives will be taken by some aggregation rule. By our Theorem 3.1, this has to be a q-majority rule. Using different q-majority rules for different pairs of alternatives is ruled out by neutrality. Hence, there must be some q-majority rule which is used to generate the social decision about each pair of alternatives.

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