Minimum Norm Solutions for Cooperative Games *

Klaus Kultti
Department of Economics, University of Helsinki,
00014 Helsinki, Finland

Hannu Salonen
Department of Economics, University of Turku,
20014 Turku, Finland
(mailing address)
e-mail: hansal@utu.fi

February 16, 2006

Abstract

A solution f for cooperative games is a *minimum norm solution*, if the space of games has a norm such that $f(v)$ minimizes the distance (induced by the norm) between the game v and the set of additive games. We show that each linear solution having the inessential game property is a minimum norm solution. Conversely, if the space of games has a norm, then the minimum norm solution w.r.t. this norm is linear and has the inessential game property. Both claims remain valid also if solutions are required to be efficient. A minimum norm solution, the least square solution, is given an axiomatic characterization.

JEL code: C71

Key words: Cooperative games, solutions, minimum norm, Banzhaf value

*We thank Hannu Nurmi for useful comments and the Yrjö Jahnsson Foundation for financial support.

1 Introduction

The games we analyze are the familiar n -person cooperative games with side payments. A solution is a function f that to each game v assigns an *additive game* $f(v)$. Additive games are like (signed) measures: each player is given a payoff, and the payoff to a coalition is just the sum of the payoffs of the players in that coalition. Additive games are also called "inessential games". A solution f has the *inessential game property*, if the solution to any additive game p is that game itself: $f(p) = p$. A solution f is *linear*, if it is additive ($f(v+w) = f(v)+f(w)$), and satisfies $f(av) = af(v)$ for all real numbers a.

A solution f is a *minimum norm solution*, if the space of games has a norm such that $f(v)$ minimizes the distance (induced by the norm) between the game v and the set of additive games. We show that each linear solution having the inessential game property is a minimum norm solution. Conversely, if the space of games has a norm, then the minimum norm solution w.r.t. this norm is linear and has the inessential game property. Both claims remain valid also if solutions are required to be efficient. We consider only norms that can be derived from some inner product.

Some of the best known solutions for cooperative games, such as the Shapley value and the Banzhaf value, satisfy linearity and have the inessential game property. Classes of solutions such as semivalues (Dubey, Neyman and Weber (1981) and least square values (Ruiz, Valenciano and Zarzuelo (1996, 1998)) satisfy these conditions as well. So all these solutions are minimum norm solutions. We characterize axiomatically one of the least square values defined by Ruiz, Valenciano and Zarzuelo (1996, 1998).

Minimum norm solutions have some intuitive appeal. All information about players' importance, influence or power is given by the function v that represents the game. The payoff $f(v)_i$ to player i should presumably reflect the importance of this player in v as closely as possible. In other words, a solution $f(v)$ should be a best possible representation of v among the additive games. Minimizing the distance between a game and the set of its possible solutions is one possible way to formalize this idea.

The plausibility of solutions is often judged by investigating the axioms the solutions satisfy. The linearity property satisfied by the minimum norm solutions has received some criticism in the literature. If we don't accept linearity, we cannot find the idea behind the minimum norm solutions very interesting either. Conversely, if linearity is accepted as a reasonable axiom, then minimum norm solutions should be accepted as well. Unless, of course, one cannot accept the inessential game property. But this is probably one of the least controversial axioms discussed in the literature.

The paper is organized in the following way. In Section 2 notation is introduced. Main results are given in Section 3. The least square solution is axiomatized in Section 4. Section 5 contains some remarks.

2 Preliminaries

Given a nonempty finite player set $N = \{1, \ldots, n\}$, an n *-person cooperative game* is a function v that assigns to each *coalition* $S \subset N$ a real number $v(S)$ with the convention $v(\emptyset) = 0$. Hence a game v may be viewed as a vector in the linear space $\mathbb{R}^{2^N-1}$. We denote by v_i the coordinate of v corresponding to the singleton coalition $\{i\}$. We denote by $\mathbb{V}$ the space $\mathbb{R}^{2^N-1}$ of all n -person games. The origin of $\mathbb{V}$ is denoted by $\bar{0}$.

An *inner product* on $\mathbb{V}$ is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ such that for all games v, v', w: (1) $\langle v, w \rangle = \langle w, v \rangle$; (2) $\langle av + bv', w \rangle = a\langle v, w \rangle + b\langle v', w \rangle$, for all $a, b \in \mathbb{R}$; (3) $\langle v, v \rangle \geq 0$; (4) $\langle v, v \rangle = 0$ implies $v = \bar{0}$. A norm $\| \cdot \|$ on

$\mathbb{V}$ can be defined from an inner product $\langle\cdot,\cdot\rangle$ by $\|v\| = \sqrt{\langle v,v\rangle}$. A metric d can be defined from a norm by $d(v,w) = \|v-w\|$. Interpreting v as a column vector, its transpose v^T is a row vector. Then the usual dot product $v^T w = \sum_{S\subset N} v(S)w(S)$ is an inner product. The norm derived from this inner product is the Euclidean norm.

A game $v \in V$ is *additive*, if $v(S) = \sum_{i\in S} v_i$, for all $S \subset N$. We denote by $\mathbb{A}$ the set of all additive n -person games, and note that it is a linear subspace of $\mathbb{V}$. A game $v \in \mathbb{V}$ is *convex*, if $v(S) - v(S\setminus\{i\}) \geq v(T) - v(T\setminus\{i\})$, for all $i \in N$ and for all coalitions S, T such that $i \in T \subset S$. A game v is *monotone*, if $S \subset T$ implies $v(S) \leq v(T)$, for all $S, T \subset N$.

A *solution* to n -person cooperative games is a function $f : \mathbb{V} \longrightarrow \mathbb{A}$ that to each game v associates an additive game $f(v)$. The best-known solution is the Shapley value ϕ:

$$\phi(v)_i = \sum_{S\ni i} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S\setminus\{i\})],$$

for all $i \in N$, where $s = |S|$ denotes the cardinality of coalition S. Let $\pi : N \longrightarrow N$ be any bijection. To each $v \in \mathbb{V}$ define a game πv by $\pi v(\pi(S)) = v(S)$, for all $S \subset N$, where $\pi(S) = \{\pi(i) \mid i \in S\}$. The Shapley value is the only solution f on $\mathbb{V}$ satisfying the following four conditions (Shapley 1953; Winter 2002). The conditions must hold for every game $v \in \mathbb{V}$, but we don't repeat this in the statements of the conditions.

A1. (Efficiency) $f(v)(N) = v(N)$.

A2. (Anonymity) $f(\pi v)_{\pi(i)} = f(v)_i$, for all $i \in N$, and any bijection $\pi : N \longrightarrow N$.

A3. (Dummy) If $v(S) - v(S \setminus \{i\}) = 0$ for all $S \subset N$, then $f(v)_i = 0$.

A4. (Additivity) $f(v + w) = f(v) + f(w)$.

Another well-known solution is the Banzhaf value β:

$$\beta(v)_i = \frac{1}{2^{n-1}} \sum_{S \ni i} [v(S) - v(S \setminus \{i\})],$$

for all $i \in N$. It satisfies all the properties mentioned above except efficiency. Both the Shapley value and the Banzhaf value satisfy the following linearity condition, which is a bit stronger than additivity.

A5. (Linearity) $f(av + bw) = af(v) + bf(w)$, for all $a, b \in \mathbb{R}$.

Both the Shapley value and the Banzhaf value satisfy the following three conditions.

A6. (Inessential game) $f(v) = v$, for all additive games v.

A7. (Coalitional mononicity) If $v(S) > w(S)$ and $v(T) = w(T)$ for all $T \neq S$, then $f(v)_i \geq f(w)_i$ for all $i \in S$.

A8. (Positivity) If v is monotone, then $f(v) \geq \bar{0}$.

The *core* $C(v)$ of a game v consists of additive games $x \in \mathbb{A}$ such that $x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subset N$. In general the core may be empty, but the Shapley value is in the core of any convex game.

3 Main results

Given an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$, consider the following problem for each $v \in \mathbb{V}$.

$$\text{Min}\langle v - p, v - p\rangle, \; s.t.\, p \in \mathbb{A}, p(N) = v(N). \tag{1}$$

This is of course the same as minimizing the norm $\|v - p\| = \sqrt{\langle v - p, v - p\rangle}$ with the same restrictions.

Theorem 1 *For each $v \in \mathbb{V}$, the solution $f(v)$ to the problem (1) exists uniquely. The function $f : \mathbb{V} \longrightarrow \mathbb{A}$ is efficient, linear and has the inessential game property.*

Proof. Choose $v \in \mathbb{V}$ arbitrarily. Let $M(v) = \{p \in \mathbb{A} \mid p(N) = v(N)\}$, and note that $M(v)$ is a linear manifold in $\mathbb{A}$. By the Classical Projection Theorem (Luenberger 1969, p.51), the programme (1) has a unique solution $f(v)$, and $f(v)$ is the only $p \in M(v)$ having the property

$$\langle v - p, q - p\rangle = 0, \text{ for all}\, q \in M(v) \tag{2}$$

The solution f is efficient by definition. If $p \in \mathbb{A}$ and $w = v + p$, then clearly $f(w) = f(v) + p$, so f has the inessential game property.

Let $w = v + v'$ for some $v' \in \mathbb{V}$. We want to show that $f(w) = f(v) + f(v')$, to establish additivity of f. So it suffices to show that equation (2) holds for all $q \in M(w)$, when v in that equation is replaced by $v + v'$ and p is replaced by $f(v) + f(v')$. Take an arbitrary $p \in M(v)$, and note that $q - p \in M(v')$, when $q \in M(w)$. Similarly, $q - p' \in M(v)$ when $q \in M(w), p' \in M(v')$. With these substitutions, the left hand side of equation (2) gets the following form:

$$\langle v + v' - f(v) - f(v'), q - f(v) - f(v')\rangle = \langle v - f(v), q - f(v) - f(v')\rangle + \langle v' - f(v'), q - f(v) - f(v')\rangle.$$

The first term of the sum on the right hand side of this equation is zero, because $q - f(v') \in M(v)$ and $f(v)$ solves (1) for v. The second term of the sum is also zero, because $q - f(v) \in M(v')$ and $f(v')$ solves (1) for v'. Therefore f is additive.

If $w = av$ for some $a \in \mathbb{R}$, then obviously $f(w) = af(v)$, and therefore f is linear. Q.E.D.

Note that if we solve the problem (1) without the efficiency constraint $p(N) = v(N)$, the solution would be the orthogonal projection $p(v)$ of v into $\mathbb{A}$ with respect to the inner product $\langle\cdot,\cdot\rangle$. This solution $p(v)$ satisfies linearity and has the inessential game property. The solution $f(v)$ of the problem (1) is then obtained by taking the orthogonal projection of $p(v)$ w.r.t. $\langle\cdot,\cdot\rangle$ into $M(v) = \{p \in \mathbb{A} \mid p(N) = v(N)\}$.

Our next result states that to any linear solution f that has the inessential game property, there is an inner product $\langle\cdot,\cdot\rangle$ such that $f(v)$ solves (1) for all games v. We need some new notation. Let $\mathbb{A}^\perp$ be the orthogonal complement of $\mathbb{A}$ with respect to the dot product. That is, $\mathbb{A}^\perp = \{v \in \mathbb{V} \mid v^T p = 0, \text{for all } p \in \mathbb{A}\}$. Any game $v \in \mathbb{V}$ can be uniquely expressed as the sum $v = v^\perp + p^v$, where $v^\perp \in \mathbb{A}^\perp$ and $p^v \in \mathbb{A}$.

Theorem 2 *Let $f : \mathbb{V} \longrightarrow \mathbb{A}$ be any efficient linear solution that has the inessential game property. Then there is an inner product $\langle\cdot,\cdot\rangle$ such that $f(v)$ solves (1) for all games $v \in \mathbb{V}$.*

Proof. A game v can be expressed as a sum $v = v^\perp + p^v$ in a unique way, $v^\perp \in \mathbb{A}^\perp$, $p^v \in \mathbb{A}$. The projection map $v \longrightarrow v^\perp$ is linear, so there is a matrix Q such that $Qv = v^\perp$ for all $v \in \mathbb{V}$.

Since f is a linear solution, it has a matrix F: $Fv = f(v)$, for all games v. Define

$$\langle v, w\rangle = v^T[F^T F + Q^T Q]w, \text{ for all games } v, w \in \mathbb{V}. \tag{3}$$

This is an inner product, since the matrix $F^T F + Q^T Q$ is symmetric and positive definite. Symmetry is clear, so let us check positive definiteness. Let v be any game such that $v(S) \neq 0$ for at least one coalition S. Then $v^T[F^T F + Q^T Q]v = f(v)^T f(v) + (v^\perp)^T v^\perp \geq 0$. The value is exactly zero only if $f(v) = \bar{0}$ and $v^\perp = \bar{0}$. If $v^\perp = \bar{0}$, then $v \in \mathbb{A}$, but in this case $f(v) = v$ since f has the inessential game property, so $f(v) \neq \bar{0}$. Therefore $v^T[F^T F + Q^T Q]v > 0$, and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{V}$.

Take any game v. Then

$$\langle v - f(v), v - f(v)\rangle = f(v - f(v))^T f(v - f(v)) + (v^\perp)^T v^\perp = (v^\perp)^T v^\perp,$$

since f is linear and has the inessential game property. If $p \neq f(v)$ is an arbitrary additive game, then

$$\langle v - p, v - p\rangle = (f(v) - p)^T(f(v) - p) + (v^\perp)^T v^\perp.$$

Hence $\langle v - f(v), v - f(v)\rangle < \langle v - p, v - p\rangle$, if $p \neq f(v)$, and $f(v)$ solves the problem (8). Q.E.D.

Note that if we don't assume that f is efficient in Theorem 2, then there is an inner product $\langle\cdot,\cdot\rangle$ such that $f(v)$ solves the problem (1) *without* the efficiency constraint $p(N) = v(N)$.

4 Least square values

Ruiz, Valenciano and Zarzuelo (1996, 1998) study least square values, which they define in the following way. Let $m \in \mathbb{V}$ be a nonnegative game (vector) such that $m(S) > 0$ for some S, and $m(S) = m(T)$ if $|S| = |T|$. Given a game $v \in \mathbb{V}$ and weights m, the least square value L^m solves the following minimization problem;

$$\text{Min} \sum_{S \subset N} m(S)(v(S) - p(S))^2, \ s.t. \ x \in \mathbb{A}, p(N) = v(N) \tag{4}$$

A solution is a least square value, if and only if it satisfies efficiency, linearity, symmetry, inessential game, and coalitional monotonicity (Ruiz, Valenciano and Zarzuelo 1998, Theorem 8, p.116). Symmetry is a slightly weaker condition than anonymity: the result is true also if symmetry is replaced by anonymity. Note that the least square solutions belong to the class of minimum norm solutions of Section 3.

If the weights for all coalitions are the same, then the solution to formula (4) minimizes the Euclidean distance between v and the set of all additive games p such that $v(N) = \sum_{i \in N} p_i$. Denote this solution by L. Ruiz, Valenciano and Zarzuelo (1998, Theorem 12) show that $L = \tilde{\beta}$, where $\tilde{\beta}$ is the additive normalization of the Banzhaf value β: $\tilde{\beta}(v)_i = \beta(v)_i + [v(N) - \beta(v)(N)]/n$, for all games v and all players $i \in N$.

To characterize L (and $\tilde{\beta}$) we need the following axiom.

A9. (Average coalitional monotonicity) If $\sum_{S \ni i} v(S) < \sum_{S \ni j} v(S)$, then $f(v)_i \leq f(v)_j$, for all $i, j \in N$.

This axiom says that the average value of coalitions containing i is less than the average value of coalitions containing j, then the solution should not give to i more payoff than to j. Let u be the *uniform game*: $u(S) = s/n$ for all $S \subset N$.

Theorem 3 *L is the only solution satisfying efficiency, additivity, inessen-*

tial game and average coalitional monotonicity.

Proof. (i) L satisfies the axioms. Since L is a least square solution, it is efficient, additive and has the inessential game property. Because $L = \tilde{\beta}$, L satisfies the average coalitional monotonicity, if and only if the Banzhaf value β satisfies this axiom as well.

The Banzhaf value β is additive, and has the inessential game property. Hence $\beta(v) = \beta(v^\perp) + p^v$. Let us calculate $\beta(v^\perp)$. Define for each $i \in N$ the additive game δ^i by $\delta^i(S) = 1$ if $i \in S$, and $\delta^i(S) = 0$ if $i \notin S$. Since $(v^\perp)^T \delta^i = 0$ and $(v^\perp)^T \delta^i = \sum_{S \ni i} v^\perp(S)$ for all i, we have that $\sum_{S \subset N} v^\perp(S) = \sum_{S \ni i}[v^\perp(S) + v^\perp(S \setminus \{i\})] = \sum_{S \ni i}[v^\perp(S \setminus \{i\})]$. Hence $-\sum_{S \ni i} v^\perp(S \setminus \{i\}) = -\sum_{S \subset N} v^\perp(S)$. So the Banzhaf value satisfies $\beta(v^\perp)_i = -2^{1-n} \sum_{S \subset N} v^\perp(S)$, *i.e.*

$$\beta(v^\perp) = \frac{-n}{2^{n-1}}[\sum_{S \subset N} v^\perp(S)]u \tag{5}$$

That is, the Banzhaf value of $v^\perp$ divides the sum $\frac{-n}{2^{n-1}}[\sum_{S \subset N} v^\perp(S)]$ evenly between players.

Let v be a game such that $\sum_{S \ni i} v(S) < \sum_{S \ni j} v(S)$. By the previous paragraph $\sum_{S \ni i} v^\perp(S) = \sum_{S \ni j} v^\perp(S) = 0$. Because $v = v^\perp + p^v$, we have that $\sum_{S \ni i} v(S) = \sum_{S \ni i} p^v(S)$. Each player i belongs to 2^{n-1}coalitions, and each $j \neq i$ belongs to half of these coalitions. But $\sum_{S \subset N} p(S) = 2^{n-1}p(N)$ for all additive games p implies

$$\sum_{S \ni i} p^v(S) = 2^{n-1}p_i^v + 2^{n-2}p^v(N \setminus \{i\}) = 2p_i^v + 2^{n-2}p(N). \tag{6}$$

Therefore $\sum_{S \ni i} v(S) < \sum_{S \ni j} v(S)$ if and only if $p_i^v < p_j^v$. Because the Banzhaf value β is additive and has the inessential game property, we have that $\beta(v) = \beta(v^\perp) + p^v$. So by equation (5), $\beta(v)_i < \beta(v)_j$ if and only if $\sum_{S \ni i} v(S) < \sum_{S \ni j} v(S)$, and β satisfies the average coalitional monotonicity.

(ii) Let f be any solution satisfying the axioms of the theorem. Let us show that $f(v^{\perp})_i = f(v^{\perp})_j$ for all players i, j. Suppose to the contrary that $f(v^{\perp})_i < f(v^{\perp})_j$ for some players i, j and some game $v^{\perp}$. For any real number $a > 0$, let p^a be the additive game such that $p^a_i = a$ and $p^a_j = 0$ for all $j \neq i$. Let $v = v^{\perp} + p^a$, and note that $a2^{n-1} = \sum_{S \ni i} v(S) > a2^{n-2} = \sum_{S \ni j} v(S)$. So by the average coalitional monotonicity, $f(v)_i \geq f(v)_j$. This implies that $f(v^{\perp})_i + a \geq f(v^{\perp})_j$, since f is additive and has the inessential game property. But since this holds for arbitrarily small positive numbers a, we have a contradiction with $f(v^{\perp})_i < f(v^{\perp})_j$.

By efficiency of f, $f(v^{\perp}) = L(v^{\perp})$ for any game $v^{\perp}$. Since both f and L are additive and have the inessential game property, we must have $f(v) = L(v)$ for all games v. Q.E.D.

The Banzhaf value β satisfies all other axioms of Theorem 3 except efficiency. The Shapley value satisfies all these axioms except average coalitional monotonicity. Define a solution f by $f(v) = 0.5L(v) + v(N)u$ for all games v. So $f(v)$ picks the midpoint of $L(v)$ and the egalitarian solution $v(N)u$. It is easy to see that f does not have the inessential game property but satisfies the remaining axioms of Theorem 3. Finally, define a solution g in the following way. If p is an additive game, the $g(p) = p$. If v is not an additive game, then let M be the set of the players $i \in N$ for whom the sum $\sum_{S \ni i} v(S)$ is the greatest, and let $g(v)_i = v(N)/|M|$ for $i \in M$ and $g(v)_i = 0$ for $i \in N \setminus M$. Clearly g is not additive but satisfies all the other axioms of Theorem 3. So none of the axioms of Theorem is implied by the remaining axioms.

5 Remarks

(1) Keane (1969) has shown that the Shapley value satisfies the minimization problem (4) for appropriately chosen weights $m(S)$. Hence the Shapley value is a least square value and therefore a minimum norm solution.

(2) It is well known that the Shapley value is always in the core of a convex game. This is not a general property of efficient minimum norm solutions as we show in the next example.

EXAMPLE 1. Let $N = \{1, 2, 3, 4\}$, and define v in the following way. For single person coalitions, $v_i = 0$ for $i < 4$, and $v_4 = 1$. If $S \subset \{1, 2, 3\}$ and $|S| > 1$, then $v(S) = -5 + 4|S|$. If $4 \in S$ and $S \neq N$, then $v(S) = v(S \setminus \{4\}) + 1$. The value of the grand coalition is $v(N) = v(\{1, 2, 3\}) + 1 + 1/4$.

The marginal contribution $v(S) - v(S \setminus \{i\})$ is nondecreasing in S for each player i, so v is convex. The Banzhaf values are $\beta(v)_i = (20 + 1/4)/8$ for $i < 4$, and $\beta_4 = (8+1/4)/8$. The least square values for players are calculated by $L(v)_i = \beta(v)_i + [v(N) - \beta(v)(N)]/4$. These values are $L(v)_i = 39/16$ for $i < 4$, and $L(v)_4 = 15/16$. Since $L(v)_4 = 15/16 < 1 = v_4$, $L(v)$ is not in the core of v.

(3) A game v is *simple*, if $v(S) = 0$ or $v(S) = 1$, for all $S \subset N$. The coalitions S such that $v(S) = 1$ are called *winning* coalitions, and the remaining coalitions are called *losing*. If v is simple, then $\sum_{S \ni i} v(S)$ is the number of those winning coalitions that include i. Solutions defined on simple games are usually called power indices, like the Shapley-Shubik index and the Banzhaf power index. We have shown that $\sum_{S \ni i} v(S) < \sum_{S \ni j} v(S)$ if and only if $\beta(v)_i < \beta(v)_j$ (see the proof of Theorem 3). So in a simple game v player i has less power than player j according to the Banzhaf power index, if and only if player i belongs to fewer winning coalitions than j. Note that

this holds also for the least square value L (or the normalized Banzhaf value $\tilde{\beta}$), since $L(v)_i = \beta(v)_i + a$ for some constant a that is the same for all players.

REFERENCES

Dubey P, Neyman A, and Weber J (1981) Value Theory without Efficiency. Mathematical Operations Research 4: 99-131

Keane M (1969) Some Topics in N -Person Game Theory. Ph.D. dissertation, Northwestern University, Evanston, IL

Luenberger D (1969) Optimization by Vector Space Methods. Wiley, New York

Ruiz LM, Valenciano F, and Zarzuelo FM (1996) The Least Square Prenucleolus and the Least Square Nucleolus. Two Values for TU Games Based on the Excess Vector. International Journal of Game Theory 25: 113-134

Ruiz LM, Valenciano F, and Zarzuelo FM (1998) The Family of Least Square Values for Transferable Utility Games. Games and Economic Behavior 24: 109-130

Shapley LS (1953) A Value for N -Person Games. Annals of Mathematic Studies 28: 307-317

Winter E (2002) The Shapley Value. Chapter 53 in Handbook of Game Theory, Vol. 3, Aumann R and Hart S (eds), Elsevier, Amsterdam