Dividing a Cake by Majority Decisions[*]

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Abstract

We consider a collective choice process where three players sequentially make proposals how to divide a given amount of resources. Afterwards one of the proposals is chosen by majority decision. If no proposal obtains a majority, a proposal is selected randomly by drawing lots. We establish the existence of the set of subgame perfect equilibria using a suitable refinement concept. In any equilibrium, the first agent offers the whole cake to the second proposal maker, who in turn offers the whole cake back to the first agent. The third agent is then indifferent between dividing the cake between him and one of the other agents.

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1 Introduction

In this paper we study the division of a cake by majority decisions and lotteries to break ties. Three players 1 through 3 propose each a division of the cake and afterwards decide on one of the suggestions by majority voting. If no proposal receives a majority of votes, the winning division is chosen by lottery to break the tie. The essential features are that there is a definite end of the collective choice process and that no resources are thrown away in case of disagreement.

There are a number of real world examples in which proposals are made sequentially and voting takes place after all the offers are seen. For instance, in legislative bargaining with simple open rules, agents are recognized sequentially for agenda setting and can make proposals, knowing all proposals made so far. If an agent brings a set of proposals to a vote and if a proposal passes, legislation ends [see Krehbiel (1991) for a survey of such legislative organizations]. Moreover, many collective decisions in committees in public or private organizations are governed by open sequential proposals which are then brought to a majority vote. The lottery rule to break ties is somewhat less common, although there are a number of real world substitutes which may be more or less close to a lottery. In many instances, as it can be the case in legislatures the chairman of a committee does not participate in the voting itself but has the right to break ties. If his preferences are not known to the committee members, his tie-breaking move can act as a lottery for the members.

We show that there exists an infinite number of subgame perfect equilibria which all yield the same two outcomes. In both outcomes, the first player making a proposal offers the whole cake to the second proposal maker, who in turn offers the whole cake back to the first player. In one outcome, the third player offers half of the cake to the first player and the rest to himself. This proposal will be chosen by a majority in the voting stage. The second outcome is symmetric to the first one. The second player gets half while the third player remains at the same share. Hence, in both outcomes the cake will merely be divided among 2 players. Player 3 is always one of the two players receiving half of the cake.

Player 3 is in a dominant position, because given any two of the former proposals, he can decide, which proposal should get a majority, by making an adequate proposal by himself. Since player 2 is aware of this, he tries always to construct his proposal in that way, that player 3 chooses him as a partner, who gets more than player 1. Finally this is the reason, why player 1 tries to hurt player 2 as much as possible by offering the whole cake to player 2, because then the only chance how player 2 can counter this proposal is to offer the whole cake to player 1 as well. The result of this competition is a situation of symmetric disadvantages for players 1 and 2 and allows player 3 to choose a partner at random. He is indifferent with whom to cooperate because he has to offer both players the same utility that compensates one of them for a tie and draws him into a coalition. Therefore, any mixed strategy of player 3 to offer half of the cake to player 1 or 2 with some probability can be played in equilibrium.

2 Relation to the Literature

Majority rule and drawing lots are standard procedures to divide resources in collective choice processes such as in legislatures or committees [e.g. Baron and Ferejohn (1989), Bernholz and Breyer (1994)]. Hence, we are interested in the positive analysis what division of resources we obtain in such cases. Our work is related to two strands of literature. First, the division of resources has been studied under alternative collective choice processes. Mueller (1978) examined the veto rule. In case of no agreement, the resources are thrown away. Equilibria in such games show a strong tendency towards equal shares for each individual. Our analysis is concerned with majority decisions and takes a view that resources are not thrown away in case of disagreement, but are subject to a tie-breaking procedure.

Second, in a dynamic context Baron and Ferejohn (1989) have examined the division of resources by the majority rule. If no agreement is reached, players can continue to make new proposals. The threat of delay and delay costs forces a division of resources which depends on whether amendments to original proposals are possible before voting takes place. The proposer receives disproportionate benefits and the number of recipients of positive benefits is a bare majority, but can also exceed it. In our case, there is a definite end of the collective choice process which forces players into an agreement or into a tie-breaking procedure.

3 The Game

We consider a game among three players denoted by i, j or $k = 1, 2, 3$ who want to divide a cake in the following way:

Firstly, in the **proposal stage** the players make sequentially open proposals about the division of the cake.

1. Player 1 proposes a division of the cake to players 2 and 3 not knowing their proposals.
2. Player 2 makes a second proposal knowing that of player 1.
3. Player 3 suggests a division knowing the offers of the two former players.

Secondly, in the **voting stage** the group selects one of these proposals[1] by majority voting. An offer receiving 2 or 3 votes will be implemented. If no majority can be found, thus if every proposal gets one vote,[2] the winning proposal is selected by drawing lots and each proposal is selected with probability $\frac{1}{3}$. Hence, the cake is either divided by majority decisions or by drawing lots.

Note that without loss of generality the labelling of players and thus the sequence of proposals by players is given exogenously.[3]

A proposal by player i is denoted by $D_i = (a_{i1}, a_{i2}, a_{i3})$. Thereby a_{i1}, a_{i2} and a_{i3} denote the shares of the cake offered by player i to players 1, 2, and 3 respectively. The resource constraint implies that $\sum_{j=1}^{3} a_{ij} = 1$, $\forall i$. Thus after every player has made his proposal, we have three proposals

$$\begin{aligned} D_1 &= (a_{11}, a_{12}, a_{13}) \\ D_2 &= (a_{21}, a_{22}, a_{23}) \\ D_3 &= (a_{31}, a_{32}, a_{33}) \end{aligned}$$

For simplicity of presentation, we rewrite the complete set of three proposals as the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

if convenient.

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[1] If two proposals coincide, they are treated as one proposal.

[2] The omission of abstentions will be explained in the characterization of equilibria

[3] We could add a prestage in which the label of an agent is selected at random. Such an additional stage would equalize expected utilities across players in case of identical utility functions. Since we are focusing on actual resource divisions we will skip the prestage.

Every player is risk neutral and and individual j derives utility[4]

$$U : [0,1] \rightarrow [0,1] \ , U(a_{ij}) = a_{ij} \tag{2}$$

if the proposal D_i is selected by majority voting. The expected utility u_j for player j in the case of drawing lots is then given by:

$$u_j = \frac{1}{3} \sum_{i=1}^{3} a_{ij} \ , j = 1, 2, 3 \tag{3}$$

In many parts of the analysis, we need the relative quality of the offers made to the specific players. Therefore we introduce the rank matrix R. In this matrix the best offer (the biggest share of the cake) a_{ij} for player j is labelled 3, the second best 2 and the worst 1. The case if two or more offers are identical, they have the same rank. For example the proposal matrices

$$A_1 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.7 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{pmatrix} A_2 = \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{pmatrix} \tag{4}$$

convert then into the rank matrices

$$R_1 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} R_2 = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \tag{5}$$

Additionally we denote the biggest share a_{ij} of the cake offered to player j with X_j, the second biggest with Y_j and the worst with Z_j.

If we label the entries of R as r_{ij} $(i, j = 1, 2, 3)$, where r_{ij} is the rank of player i's proposal made to player j, the conversion function $\Phi[A] = R$ is technically given by:

$$\Phi : r_{ij} = \sum_{k=1}^{3} \Theta(a_{ij} - a_{kj}) \tag{6}$$

where $\Theta(p - q)$ $(p, q \in \mathbb{R})$ is the Heavyside function given by:

$$\Theta(p - q) = \begin{cases} 1 & p - q \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

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[4] Since all players are identical, we drop the index at the utility function.

4 Voting Equilibria and Proposal Making

In this section, we derive first the voting equilibria, given a set of proposals. The voting strategy of player i is to select one proposal.[5] We first note the concept of Nash equilibrium is insufficient for voting games with non-unanimity voting rules as it can entail players using weakly dominated strategies. For that purpose we use the following refinements and tie-breaking rules (for cases of indifference between payoffs) which hold throughout the paper.

4.1 Refinements

Refinement 1
A Nash equilibrium of the voting game has to be trembling hand perfect.[6]

Trembling hand perfection has two immediate consequences.

Lemma 1
Suppose that refinement 1 holds. Then

(i) A player never votes for the least favorable proposal made to him, i.e. a proposal labeled 1 in the rank matrix.

(ii) In any Nash equilibrium where agents i and j $(i \neq j)$ vote for the same proposal, agent k $(k \neq i,\, j)$ votes for his best proposal.

The elimination of weakly dominated is standard. The second property also follows directly from the definition of trembling hand perfection as any error of players i or j leads either to a tie-break or a majority win of the best proposal of voter i.

The next refinement eliminates voting equilibria that are payoff dominated.

Refinement 2
If only one proposal includes 2 maxima, it is the unique equilibrium.

This property follows from payoff dominance (see Fudenberg and Tirole 1992). A proposal with 2 maxima for say individuals i and j is a Nash equilibrium supported by the votes of i and j. Any other possible Nash equilibrium is worse off for i and j and hence payoff dominated for the coalition $\{i, j\}$.

Even if both refinements are applied, we will have multiple equilibria. For that purpose, we use the notion of correlated equilibria with public randomization introduced by Aumann (1974) (see e.g. Myerson 1991 for discussion). This concept assumes that voters engage in preplay communication and use a coordination device in order to settle

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[5] Allowing abstention does not change the voting equilibria. Abstention is weakly dominated by voting for the best proposal for player i.

[6] See Selten (1975) for the original formalization and Fudenberg and Tirole (1992) for a survey.

on a particular equilibrium. Such a device is a publicly observable random variable which agents use to determine which equilibrium should be played. For instance, agents may flip a coin or a mediator announces the outcome of the randomization process.

This is illustrated in the following example:

$$A = \begin{pmatrix} 0.6 & 0.0 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.2 & 0.8 & 0.0 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \tag{8}$$

In this case D_1 and D_2 are equilibria and Player 1 and Player 3 prefer their second best offer to a tie-break. Furthermore these proposals contain the first and second best offer of Player 1 and Player 3. Therefore, we assume that Player 1 and Player 3 play a correlated strategy. With probability $p_1 = \frac{1}{2}$ both play D_1 and with $p_2 = \frac{1}{2}$ they play D_2.

Accordingly we use the concept of correlated equilibria:

Refinement 3
A correlated equilibrium arises if two proposals D_i, D_j $(i \neq j,\ i,j = 1,2,3)$ are equilibria and they contain the best and second best offer of the same two players. In the correlated equilibrium both players will under consideration vote with probability $\frac{1}{2}$ for D_i or D_j. The expected payoffs for all players are then given by $C_{ij} = \frac{1}{2}(D_i + D_j)$.

We note that correlated equilibria cannot be used in particular cases, when two equilibria occur, i.e.:

$$A = \begin{pmatrix} 0.0 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \tag{9}$$

Now D_2 and D_3 are equilibria, but Player 2, who is needed for the majority of D_3, has no incentive to establish a coordination for D_2 and D_3, because D_3 contains his second best and D_2 his worst offer. Two these cases are handled is discussed in subsection 4.3.

4.2 Tie-breaking rules

In this subsection we introduce some tie-breaking rules which simplify the exposition. Since agents maximize the expected share of the cake they receive, there is only one possible motive for an agent to deviate from voting for the best proposal and vote for the second best offer instead: He cannot establish a majority of votes for his best proposal, but the player prefers his second best offer to a tie-break. To illustrate this case, consider the following example involving the proposal matrix

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 \\ 0.5 & 0.5 & 0.0 \\ 0.0 & 0.1 & 0.9 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \tag{10}$$

Because of the refinement 1 player 2 votes for D_2 and player 3 for D_3. D_1, favored by player 1, has, thus, no chance to become elected, but $Y_1 = \frac{1}{2}$ is greater than $u_1 = \frac{1}{3}(\frac{7}{10} + \frac{1}{2} + 0) = \frac{2}{5}$, his expected share in case of drawing lots. Player 1, therefore, votes for D_2 together with player 2 in order to avoid a tie-break decision. The case, when the payoffs of the second-best offer and drawing lots coincide, is resolved in the following tie-breaking rule:

Tie-Break Rule 1
All players prefer their second best bid to drawing lots if $Y_j = u_j$.

Note that "prefer" in tie-breaking rule 1 means that player i will avoid drawing lots, if he is indifferent between his second best proposal and drawing lots. The tie-breaking rule means that player i will vote for his best proposal, if it receives a majority or he will vote for the second-best proposal. The tie-breaking rule immediately implies

Lemma 2
Suppose that there exists at least one voting equilibrium where a majority is formed. Then, drawing lots will not occur as an equilibrium.

To formulate the next tie-breaking rule, we denote an equilibrium as a single proposal equilibrium, if a single proposal is selected. A single proposal equilibrium necessarily requires, that pure voting strategies are played and a majority supports on proposal.

Tie-Break Rule 2
A player prefers a single proposal equilibrium to a correlated equilibrium and a correlated equilibrium to drawing lots, if the payoffs coincide.

4.3 Proposal Making

Having characterized the structure of voting equilibria, we turn to proposal making. To formulate the proposal making stage, we face two problems:

Firstly multiple voting equilibria may still exists (i.e. see the proposal matrix in equation 9). Secondly players may be indifferent between several proposals (i.e. player 3 is always indifferent between at least two proposals, if $D_1 = (a_{11}, 1 - a_{11}, 0)$ and $D_2 = (1 - a_{11}, a_{11}, 0)$).

Addressing the first point, we introduce another refinement:

Refinement 4
Given the proposals D_1 *and* D_2*, player 3 makes his proposal such, that the proposal matrix* A *exhibits a single proposal equilibrium, a correlated equilibrium or an equilibrium with drawing lots.*[7]

This refinement ensures, that payoffs in the voting stage are well-defined. This can be justified by aversion against strategic uncertainty.

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[7] We will show, that this is always possible for player 3 (see proof of the overall equilibrium).

The second point is handled by an additional tie-breaking rule

Tie-Break Rule 3
If player i is indifferent between making several proposals, every proposal is submitted with the same probability.

4.4 The Structure of Voting Equilibria

After these preparations, we provide an overview about the structure of voting equilibria. We start with calculating equilibria in two examples for. For that purpose we extend the conversion of the proposal matrix A into the rank matrix R, since often it is necessary to distinguish, whether it is possible for a player to deviate to his second-best proposal or not. This is done by introducing the rank 2^*, if any $Y_j \geq u_j$ $(j = 1, 2, 3)$. If we consider the example given in (10), we get

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 \\ 0.5 & 0.5 & 0.0 \\ 0.0 & 0.1 & 0.9 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \tag{11}$$

For calculating the equilibria we ask, when player i is deviating given the votes of players j and k $(i \neq j \neq k)$. If player i is not deviating his payoff is underlined.

$P_1 \longrightarrow D_1$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	$\underline{0.7}$ 0.0 0.3	$\underline{0.7}$ 0.0 0.3	$\underline{0.7}$ 0.0 $\underline{0.3}$
D_2	$\underline{0.7}$ $\underline{0.0}$ 0.3	$\underline{0.5}$ $\underline{0.5}$ 0.0	0.4 $\underline{0.2}$ $\underline{0.4}$
D_3	$\underline{0.7}$ 0.0 0.3	0.4 0.2 0.4	$\underline{0.0}$ 0.1 $\underline{0.9}$

$P_1 \longrightarrow D_2$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	0.7 0.0 0.3	0.5 0.5 0.0	0.4 0.2 $\underline{0.4}$
D_2	0.5 $\underline{0.5}$ 0.0	0.5 $\underline{0.5}$ 0.0	$\underline{0.5}$ $\underline{0.5}$ $\underline{0.0}$
D_3	0.4 0.2 0.4	$\underline{0.5}$ 0.5 0.0	0.0 0.1 $\underline{0.9}$

$P_1 \longrightarrow D_3$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	0.7 0.0 0.3	0.4 0.2 0.4	0.0 0.1 $\underline{0.9}$
D_2	0.4 $\underline{0.2}$ 0.4	0.5 $\underline{0.5}$ 0.0	0.0 $\underline{0.1}$ $\underline{0.9}$
D_3	0.0 0.1 0.9	0.0 0.1 0.9	0.0 0.1 $\underline{0.9}$

In this case only the equilibrium voting scheme is

$$\left.\begin{array}{l} P_1 \longrightarrow D_2 \\ P_2 \longrightarrow D_2 \\ P_3 \longrightarrow D_3 \end{array}\right\} \Longrightarrow D_2 \text{ is chosen} \tag{12}$$

We see, that these calculations can be done, by just concerning the extended rank matrix. For that purpose, we look at the example given in (8). We get the extended rank matrix

$$A = \begin{pmatrix} 0.6 & 0.0 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.2 & 0.8 & 0.0 \end{pmatrix} \longrightarrow R = \begin{pmatrix} 3 & 1 & 2^* \\ 2^* & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \tag{13}$$

an end up with

$$P_1 \longrightarrow D_1$$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	$\underline{3}$ 1 2^*	$\underline{3}$ 1 $\underline{2}^*$	$\underline{3}$ 1 2^*
D_2	$\underline{3}$ 1 2^*	$\underline{2}^*$ 2 $\underline{3}$	T T T
D_3	$\underline{3}$ $\underline{1}$ $\underline{2}^*$	T $\underline{T}$ T	$\underline{1}$ $\underline{3}$ 1

$$P_1 \longrightarrow D_2$$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	3 1 2^*	2^* 2 $\underline{3}$	T T T
D_2	2^* 2 3	2^* 2 $\underline{3}$	$\underline{2}^*$ 2 3
D_3	T $\underline{T}$ T	$\underline{2}^*$ $\underline{2}$ $\underline{3}$	1 $\underline{3}$ 1

$$P_1 \longrightarrow D_3$$

$P_2 \diagdown P_3$	D_1	D_2	D_3
D_1	3 1 $\underline{2}^*$	T T T	1 3 1
D_2	T T T	2^* 2 $\underline{3}$	1 3 1
D_3	1 $\underline{3}$ 1	1 $\underline{3}$ $\underline{1}$	1 $\underline{3}$ 1

Now two voting schemes are equilibria

$$\left.\begin{array}{l} P_1 \longrightarrow D_1 \\ P_2 \longrightarrow D_3 \\ P_3 \longrightarrow D_1 \end{array}\right\} \Longrightarrow D_1 \text{ is chosen} \quad \left.\begin{array}{l} P_1 \longrightarrow D_2 \\ P_2 \longrightarrow D_3 \\ P_3 \longrightarrow D_2 \end{array}\right\} \Longrightarrow D_2 \text{ is chosen} \tag{14}$$

Since in both voting schemes player 1 and 3 are forming the majority, these equilibria are correlated.

In the following we describe the possible extended rank matrices and their equilibria. When a single proposal equilibrium, an equilibrium with drawing lots or a correlated arise, they are denoted by D_i^*, C_{ij}^* and L^* respectively. The treated rank matrices R are thereby to be understood as a representatives for a classes of matrices derivable by interchanging rows and columns of R or exchanging two coefficients of the matrix.

For example

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

is amongst others a representative of

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Furthermore we omit, when convenient, cases with $X_i \neq Y_i = Z_i$, because they are qualitatively equal to cases with $X_i \neq Y_i \neq Z_i \wedge Y_i < u_i$ and we use $r_{ij} = 2^{(*)}$ if the conversion of a_{ij} in 2 or 2^* does not change the equilibrium outcome.

1. $D_1 = D_2 = D_3 \longrightarrow R$ shrinks to a (3×1) matrix and we have

$$\begin{pmatrix} 3 & 3 & 3 \end{pmatrix} \longrightarrow D_1^* \tag{15}$$

2. $D_1 \neq D_2 = D_3 \longrightarrow R$ shrinks to a (3×2) matrix and we have

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \end{pmatrix} \longrightarrow \left\{\begin{array}{l} D_1^* \\ D_2^* \end{array}\right. \quad \begin{pmatrix} 3 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \longrightarrow D_1^* \tag{16}$$

3. $D_1 \neq D_2 \neq D_3$

(a) $X_1 = Y_1 = Z_1 \wedge X_2 \neq Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 3 & 1 \\ 3 & 2^{(*)} & 2^{(*)} \\ 3 & 1 & 3 \end{pmatrix} \longrightarrow \left\{ \begin{array}{l} D_1^* \\ D_3^* \end{array} \right. \tag{17}$$

(b) $X_1 = Y_1 \neq Z_1 \wedge X_2 = Y_2 \neq Z_2 \wedge X_3 = Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \longrightarrow \left\{ \begin{array}{l} D_1^* \\ D_2^* \\ D_3^* \end{array} \right. \tag{18}$$

(c) $X_1 = Y_1 \neq Z_1 \wedge X_2 = Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2^{(*)} \end{pmatrix} \longrightarrow \left\{ \begin{array}{l} D_1^* \\ D_2^* \end{array} \right. \qquad \begin{pmatrix} 3 & 1 & 2^{(*)} \\ 3 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \longrightarrow \left\{ \begin{array}{l} D_2^* \\ D_3^* \end{array} \right. \tag{19}$$

(d) $X_1 = Y_1 \neq Z_1 \wedge X_2 \neq Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

$$\begin{gathered} \begin{pmatrix} 3 & 1 & 3 \\ 3 & 3 & 1 \\ 1 & 2^{(*)} & 2^{(*)} \end{pmatrix} \longrightarrow \left\{ \begin{array}{l} D_1^* \\ D_2^* \end{array} \right. \qquad \begin{pmatrix} 3 & 1 & 2^{(*)} \\ 3 & 3 & 1 \\ 1 & 2^{(*)} & 3 \end{pmatrix} \longrightarrow D_2^* \\ \begin{pmatrix} 3 & 1 & 2^{(*)} \\ 3 & 2^{(*)} & 1 \\ 1 & 3 & 3 \end{pmatrix} \longrightarrow D_3^* \end{gathered} \tag{20}$$

(e) $X_1 \neq Y_1 \neq Z_1 \wedge X_2 \neq Y_2 \neq Z_2 \wedge X_3 \neq Y_3 \neq Z_3$

i. Doublemax:=$\big(\exists\, r_{ij} = r_{ik} = 3 \quad j \neq k \quad i, j, k \in \{1, 2, 3\}\big)$

$$\begin{pmatrix} 3 & 3 & 2 \\ 1 & 2^{(*)} & 3 \\ 2^{(*)} & 2 & 1 \end{pmatrix} \longrightarrow D_1^* \tag{21}$$

ii. $Y_1 < u_1 \wedge Y_2 < u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow L^* \tag{22}$$

iii. $Y_1 \geq u_1 \wedge Y_2 < u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2 \\ 2^* & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow D_2^* \tag{23}$$

iv. $Y_1 \geq u_1 \wedge Y_2 \geq u_2 \wedge Y_3 < u_3 \wedge$ no Doublemax

$$\begin{gathered} \begin{pmatrix} 1 & 3 & 2 \\ 2^* & 1 & 3 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow \begin{cases} D_2^* \\ D_3^* \end{cases} \qquad \begin{pmatrix} 1 & 1 & 3 \\ 2^* & 3 & 2 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow C_{23}^* \\ \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 2 \\ 2^* & 2^* & 3 \end{pmatrix} \longrightarrow D_3^* \end{gathered} \tag{24}$$

v. $Y_1 \geq u_1 \wedge Y_2 \geq u_2 \wedge Y_3 \geq u_3 \wedge$ no Doublemax

$$\begin{pmatrix} 1 & 3 & 2^* \\ 2^* & 1 & 3 \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow \begin{cases} D_1^* \\ D_2^* \\ D_3^* \end{cases} \qquad \begin{pmatrix} 1 & 1 & 3 \\ 2^* & 3 & 2^* \\ 3 & 2^* & 1 \end{pmatrix} \longrightarrow C_{23}^* \tag{25}$$

5 Division of the Cake

In this section, we derive the overall equilibrium of the game. For that purpose we have to compare the expected payoffs of the different players making different proposals. Therefore we denote the expected payoff of player i by π_i $(i = 1, 2, 3)$.

5.1 Overall Equilibrium

The solution of the game is given by the following theorem:

Theorem 1
There exist two overall equilibria of the game:

$$\begin{array}{rcl} D_1 &=& (0,1,0) \\ D_2 &=& (1,0,0) \\ D_3 &=& (\frac{1}{2},0,\frac{1}{2}) \end{array} \quad \textit{or} \quad \begin{array}{rcl} D_1 &=& (0,1,0) \\ D_2 &=& (1,0,0) \\ D_3 &=& (0,\frac{1}{2},\frac{1}{2}) \end{array} \tag{26}$$

where $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ is a single proposal equilibrium supported by player 1 and 3 and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ is a single proposal equilibrium supported by player 2 and 3. Since player 3 is indifferent between $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ he makes each proposal with probability $\frac{1}{2}$ (Tie-Break Rule 3). This implies for the payoffs

$$\pi_1 = \frac{1}{4} \quad \pi_2 = \frac{1}{4} \quad \pi_3 = \frac{1}{2}$$

The proof of the theorem follows directly from corollary 1 and propositions 5 and 6 given in subsections 5.3, 5.4 and 5.5.

This theorem can be motivated as follows. If player 1 makes any proposal $D_1 \neq (0, 1, 0)$ player 2 counters this with a proposal D_2 such that π_1 is less than $\frac{1}{4}$. But with proposing $D_1 = (0, 1, 0)$ player 1 makes player 2 so unattractive that any counter proposal $D_2 \neq (1, 0, 0)$ will give player 2 a payoff π_2 less than $\frac{1}{4}$. This implies, that in the end the cake is divided half-by-half between player 1 and 3 or player 2 and 3.

5.2 Strategy of the Proof

We proof theorem 1 by backward induction. This is illustrated in the following steps:

- Step 1: We determine the best reaction of player 1, given any D_1 and D_2 and calculate the minimum payoff π_3 given specific relations between D_1 and D_2.
- Step 2: Given $D_1 \neq (0, 1, 0)$ we determine a proposal D_2 giving player 2 a payoff π_2, such that together with the minimum payoff π_3 of player 3 from step 1 the resource constraint implies $\pi_1 < \frac{1}{4}$ for player 1.
- Step 3: Given $D_1 = (0, 1, 0)$, we calculate the payoff π_2, if player 2 makes any proposal $D_2 \neq (1, 0, 0)$ and compare this with his payoff π_2 while making $D_2 = (1, 0, 0)$.

5.3 Strategy of Player 3

Given D_1, D_2 Player 3 has the ambition to maximize his share of the cake by making an appropriate proposal D_3.

Generally player 3 considers four possible equilibrium outcomes while constructing his proposal D_3, given D_1 and D_2.

I: D_3 is a single proposal equilibrium.

II: D_1 or D_2 is a single proposal equilibrium.

III: A correlated equilibrium arises.

VI: The cake is divided by drawing lots.

Without a proof, we give examples, where the different construction principles (I-IV) are best reactions of player 3:

$$\text{I} \quad : \quad \left.\begin{array}{c} D_1 = (0.7, 0.3, 0.0) \\ D_2 = (0.3, 0.5, 0.2) \end{array}\right\} \Longrightarrow D_3 = (0.0, 0.4, 0.6) \qquad \Longrightarrow D_3^*$$

$$\text{II[8]} \quad : \quad \left.\begin{array}{c} D_1 = (0.1, 0.3, 0.6) \\ D_2 = (0.5, 0.5, 0.0) \end{array}\right\} \Longrightarrow D_3 = (1.0, 0.0, 0.0) \qquad \Longrightarrow D_1^*$$

$$\text{III} \quad : \quad \left.\begin{array}{c} D_1 = (0.0, 0.9, 0.1) \\ D_2 = (0.4, 0.1, 0.5) \end{array}\right\} \Longrightarrow D_3 = (0.2, 0.0, 0.8) \qquad \Longrightarrow C_{23}^*$$

$$\text{IV} \quad : \quad \left.\begin{array}{c} D_1 = (0.1, 0.1, 0.8) \\ D_2 = (0.6, 0.4, 0.0) \end{array}\right\} \Longrightarrow D_3 = (0.0, 0.7 + \epsilon, 0.3 - \epsilon) \quad \Longrightarrow L^*$$

with $\epsilon > 0$ and infinitesimally small.

Which kind of the equilibrium outcome I–IV player 3 prefers, depends strongly on the relations between the proposals D_1 and D_2 characterized by the submatrix a of the proposal matrix A, which is defined by

$$A = \left(\begin{array}{ccc} \multicolumn{2}{c}{\text{a}} & a_{13} \\ & & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) \qquad \Longrightarrow \qquad a = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \tag{27}$$

and the corresponding rank matrix ρ which is calculated from a in a similar way as the R from A:

$$\rho = \phi[a] = \left(\begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right)$$

---

[8] In this case we have an continuum of best proposals $D_3 = (a_{31}, a_{32}, 1 - a_{31} - a_{32})$ with $a_{31} \in (0.9, 1]$ and $a_{32} \in [0, 1 - a_{31}]$.

where ϕ is defined similar to Φ (see (6)), with

$$\phi : \rho_{ij} = \sum_{k=1}^{2} \Theta(a_{ij} - a_{kj}) \tag{28}$$

The relevance of a and ρ will be shown in the proofs of the following propositions and corollaries.

Proposition 1
Given D_1 and D_2 and ρ is symmetric $\Longrightarrow$ determining the best reaction of player 3 implies $\pi_3 \geq \frac{3}{8}$.

Corollary 1
Given $D_1 = (0, 1, 0)$ and $D_2 = (1, 0, 0)$ $\Longrightarrow$ player 3's best proposal is $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ or $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$, which implies $\pi_1 = \pi_2 = \frac{1}{4}$ and $\pi_3 = \frac{1}{2}$.

Proposition 2
Given D_1, D_2, $a \neq \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and ρ is non-symmetric $\Longrightarrow$ determining the best reaction of player 3 implies $\pi_3 > \frac{1}{4}$.

Corollary 2
Given $a = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \vee \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ $\Longrightarrow$ player 3's best proposal is $D_3 = (\frac{1}{2} + \epsilon, \frac{1}{4}, \frac{1}{4} - \epsilon)$ or $D_3 = (\frac{1}{4}, \frac{1}{2} + \epsilon, \frac{1}{4} - \epsilon)$ which implies $\pi_1 = \pi_2 = \frac{7}{16} + \frac{\epsilon}{4}$ and $\pi_3 = \frac{1}{8} - \frac{\epsilon}{2}$.

(The proofs are given in the appendix)

5.4 Strategy of Player 2

In this section, we analyze possible reactions of player 2 given D_1 and given the derived reaction of player 3 in section 5.3.

For that purpose, we divide the proposal set of D_1 given by:

$$\mathcal{P} = \{(a_{11}, a_{12}) | (a_{11}, a_{12}) \in [0, 1] \times [0, 1 - a_{11}]\} \tag{29}$$

into four subsets:

$$\begin{aligned} \mathcal{A} &= \left[\tfrac{1}{2}, 1\right] \times [0, 1 - a_{11}] \setminus \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \\ \mathcal{B} &= \left[0, \tfrac{1}{2}\right] \times [0, 1 - a_{11}] \setminus (0, 1) \\ \mathcal{C} &= \left[\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left[\tfrac{1}{2}, \tfrac{1}{2}\right) \\ D^* &= (0, 1) \end{aligned} \tag{30}$$

Proposition 3
$\forall\ D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal D_2^s of player 2, such that ρ is symmetric and $\pi_2 > \frac{3}{8}$.

Corollary 3
If for $D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal D_2^{ns} of player 2, such that ρ is non-symmetric and $\pi_2(D_2^{ns}) \geq \pi_2(D_2^s) \Longrightarrow \pi_2(D_2^{ns}) \geq \frac{1}{2}$ or $\pi_1(D_2^{ns}) < \frac{1}{4}$.

Proposition 4
$\forall\ D_1 \in \mathcal{C}\ \exists$ a proposal D_2^{ns} of player 2, such that ρ is non-symmetric and $\pi_2 \geq \frac{1}{2}$ or the best proposal D_2^{ns} with ρ non-symmetric for player 2 implies $\pi_1 = 0$.

Corollary 4
If for $D_1 \in \mathcal{C}\ \exists$ a proposal D_2^s of player 2, such that ρ is symmetric and $\pi_2(D_2^s) \geq \pi_2(D_2^{ns}) \Longrightarrow \pi_2(D_2^s) > \frac{3}{8}$.

Proposition 5
The best reaction of player 2 given $D_1 = (0, 1, 0)$ is $D_2 = (1, 0, 0)$

(The proofs are given in the appendix)

5.5 Strategy of Player 1

Since player 1 anticipates the reactions of players 2 and 3, we obtain the following proposition:

Proposition 6
The best proposal of player 1 is $D_1 = (0, 1, 0)$.

Proof of proposition 6:

(i) Suppose $D_1 \neq (0, 1, 0)$ and D_2 with ρ is symmetric is the best reaction of player 2 $\Longrightarrow \pi_3 \geq \frac{3}{8}$ (proposition 1) and $\pi_2 > \frac{3}{8}$ (proposition 3 and corollary 3)
$\Longrightarrow$ together with the resource constraint we obtain $\pi_1 + \overset{>\frac{3}{8}}{\pi_2} + \overset{\geq\frac{3}{8}}{\pi_3} = 1 \Longrightarrow \pi_1 < \frac{1}{4}$.

(ii) Suppose $D_1 \neq (0, 1, 0)$ and D_2 with ρ is non-symmetric is the best reaction of player 2
$\Longrightarrow \pi_3 > \frac{1}{4}$ (proposition[9] 2) and $\pi_2 \geq \frac{1}{2} \vee \pi_1 < \frac{1}{4}$ (corollary 3 and proposition 4)
$\Longrightarrow$ together with the resource constraint we obtain $\pi_1 + \overset{\geq\frac{1}{2}}{\pi_2} + \overset{>\frac{1}{4}}{\pi_3} = 1 \Longrightarrow \pi_1 < \frac{1}{4}$, or we have directly $\pi_1 > \frac{1}{4}$.

$\Longrightarrow$ (i) and (ii) together with corollary 1 and proposition 5 imply that $D_1 = (0, 1, 0)$ is the best proposal of player 1.

■

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[9] Note that neither $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ is the best reaction of player 2 given $D_1 = (0, 0, 1)$ nor $D_2 = (0, 0, 1)$ is the best reaction of player 2 given $D_1 = (\frac{1}{2}, \frac{1}{2}, 0)$.

6 Discussion and Conclusion

We have examined a common collective choice process to study the allocation of resources among a group of people. The analysis reveals that the first two agents want to make each other as unattractive as possible when the third agent makes a proposal. To do so they offer each other the whole cake and the third player can ensure that he obtains one half of the cake.

This outcome exhibits a very powerful last mover advantage, whereas the other players are forced to make strategic proposals involving zero resources for themselves that they can expect one quarter of the cake. Additionally it seems surprisingly, that the first player can totally outweigh the second mover advantage of the second player. How these characteristics extend to group decisions with a larger number of individuals is an important avenue of future research. Of course there are a variety of game-theoretic considerations and alternative refinement concepts that can be examined. How robust our main findings is with regard to such extensions remains to be explored.

7 Appendix

We already mentioned that the specific relations between D_1 and D_2 are relevant for the construction of the best reaction D_3 given by $'r$ and a. Furthermore, we need a more detailed specification of the relation between the entries of a. Therefore we define

$$\begin{aligned} \mu_j &= \tfrac{1}{2}(a_{1j} + a_{2j}) \\ \overline{\mu} &= \max\{\mu_1\,, \mu_2\} \\ \underline{\mu} &= \min\{\mu_1\,, \mu_2\} \\ x_j &= \max\{a_{1j}\,, a_{2j}\} \\ y_j &= \min\{a_{1j}\,, a_{2j}\} \\ \underline{x} &= \min\{x_1\,, x_2\} \end{aligned}$$

7.1 Proof of Propositions 1 and 2 and Corollaries 1 and 2

Before we start, note that the best reaction D_3 is not always unique (i.e. $x_1 = x_2 \wedge y_1 = y_2$ or see footnote 8), but since we only want to determine the minimum share of the cake of player 3, it is sufficient to give only one best reaction.

7.1.1 Proof of Proposition 1

There are three different possibilities for ρ to be symmetric:

$$\rho_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \rho_{1'} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \tag{31}$$

For ρ_2 we have $D_1 = D_2$, and the best proposal D_3 is given by:

1. If $D_1 = (0,0,1) \Rightarrow D_3 = (0,0,1)$
2. If $D_1 \neq (0,0,1) \Rightarrow D_3 = (\underline{x}+\epsilon, 0, 1-\underline{x}-\epsilon)$ if $x_1 = \underline{x}$ and $D_3 = (0, \underline{x}+\epsilon, 1-\underline{x}-\epsilon)$ if $x_2 = \underline{x}$.

For ρ_1 or $\rho_{1'}$ it is sufficient to analyze only ρ_1, since the same arguments follow for $\rho_{1'}$ by exchanging columns in matrix a.

For ρ_1 A is given by:

$$A = \begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} a_{13} &= 1 - x_1 - y_2 \\ a_{23} &= 1 - x_2 - y_1 \\ a_{33} &= 1 - a_{31} - a_{32} \end{aligned} \tag{32}$$

In order to calculate the best reaction of player 3, the following properties of a are relevant:

(i) $x_i > a_{3i} \geq \mu_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 1, \Phi(a_{3i}) = 2^*$

(ii) $y_i > a_{3i} \geq 2y_i - x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 2, \Phi(a_{3i}) = 1$

(iii) $a_{3i} = x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 1, \Phi(a_{3i}) = 3$

With these properties, we distinguish six different kinds of proposals D_3, which will be discussed in detail below:

$$D_3^{\underline{\mu}} \qquad D_3^{\overline{\mu}} \qquad D_3^{\mu_i} \qquad D_3^{\underline{x}} \qquad D_3^{Cor} \quad D_3^{L}$$

$D_3^{\underline{\mu}}$: Player 3 offers player 1 or 2 $\underline{\mu}$ and the rest to himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \underline{\mu}$ (i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \underline{\mu} & 0 & 1-\underline{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2^* & 1 & 3 \end{pmatrix} \tag{33}$$

$D_3^{\overline{\mu}}$: Player 3 offers player 1 or 2 $\overline{\mu}$ and the rest to himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \overline{\mu}$ (i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ 0 & \overline{\mu} & 1-\overline{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2^* & 3 \end{pmatrix} \tag{34}$$

$D_3^{\mu_i}$: Player 3 offers player i $a_{3i} = \mu_i$, player j $a_{3j} = 2y_j - x_j + \epsilon$ and the rest to himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - (\mu_i + 2y_j - x_j + \epsilon)$ $(i, j = 1, 2$ $i \neq j$ and $2y_j - x_j \geq 0)$ (i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \mu_1 & 2y_2 - x_2 + \epsilon & 1 - (\mu_1 + 2y_2 - x_2 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2^* & 1 & 3 \end{pmatrix} \tag{35}$$

$D_3^{\underline{x}}$: Player 3 offers player 1 or 2 $\underline{x}$ and the rest to himself, such that D_3 is a unique equilibrium and he receives a payoff $\pi_3 = 1 - \underline{x}$ (i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \underline{x} & 0 & 1-\underline{x} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \tag{36}$$

D_3^{Cor}: Player 3 offers player i $a_{3i} = \mu_i$, player j

$$a_{3j} = \begin{cases} 2y_j - x_j + \epsilon & \text{if} \quad 2y_j - x_j \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (i, j = 1, 2 \; i \neq j)$$

and the rest to himself, such that D_3 is part of the correlated equilibrium and he receives a payoff

$$\pi_3 = \begin{cases} \frac{1}{2}(1 - (\mu_i + 2y_j - x_j) + \epsilon + a_{i3}) & \text{if} \quad 2y_j - x_j \geq 0 \\ \frac{1}{2}(1 - \mu_i + a_{i3}) & \text{otherwise} \end{cases} \qquad (i, j = 1, 2 \; i \neq j)$$

(i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ \mu_1 & 2y_2 - x_2 + \epsilon & 1 - (\mu_1 + 2y_2 - x_2 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 2^* \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix} \qquad (37)$$

D_3^{L}: Player 3 offers both other players

$$a_{3i} = \begin{cases} 2y_i - x_i + \epsilon_i & \text{if} \quad 2y_i - x_i \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, 2) \qquad (\epsilon_1, \epsilon_2 > 0 \text{ and infinitesimally small})$$

and the rest to himself, such that the proposal is selected by drawing lots and he receives a payoff

$$\pi_3 = \begin{cases} 1 - (y_1 + y_2) - \frac{1}{3}(\epsilon_1 + \epsilon_2) & \text{if} \quad 2y_i - x_i \geq 0 \quad (i = 1, 2) \\ 1 - \frac{1}{3}(3y_i + x_j + y_j + \epsilon) & \text{if} \quad \begin{cases} 2y_i - x_i \geq 0 \\ 2y_j - x_j < 0 \end{cases} \qquad (i, j = 1, 2 \; i \neq j) \\ 1 - \frac{1}{3}(x_1 + x_2 + y_1 + y_2) & \text{otherwise} \end{cases}$$

(i.e):

$$\begin{pmatrix} x_1 & y_2 & a_{13} \\ y_1 & x_2 & a_{23} \\ 0 & 2y_2 - x_2 + \epsilon_2 & 1 - (2y_2 - x_2 + \epsilon_2) \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix} \qquad (38)$$

Note that constructing D_3 such that D_1 or D_2 are unique equilibria or D_1 and D_2 form a correlated equilibrium can never be better than $D_3^{\underline{x}}$, since $1 - \underline{x} \geq a_{i3}$ $(i = 1, 2)$ and offering $D_3^{\underline{x}}$ is always possible for player 3 if ρ symmetric. This ensures, that refinement 4 can always be satisfies and players 3's best reaction is an element of $\{D_3^{\underline{\mu}}, D_3^{\overline{\mu}}, D_3^{\mu_i}, D_3^{\underline{x}}, D_3^{Cor}, D_3^{T}\}$

In order to maximize π_3 we calculate the feasible sets of the different proposals, since their feasibility depends on the relation between the a_{i3} $(i = 1, 2, 3)$ and compare the different payoffs if they intersect.

While comparing payoffs, we often use principle of the proof with contradiction. Therefore we introduce the sign ↯ to indicate a conclusion, which is contradicting the assumptions.

Since every feasible set is bounded by linear inequalities we can calculate $\min\{\pi_3\}$ given the best reaction of player 3 by a simplex algorithm.[10]

1. $D_3^{\underline{\mu}}$

   Constraints for $D_3^{\underline{\mu}}$ to be the best proposal of player 3:

1.	μ_1	$\leq$	μ_2	
2.	$2y_2$	$<$	x_2	
3.	a_{i3}	$\leq$	a_{j3}	$\Big\}\ i,j=1,2\ i\neq j$
4.	$1-\mu_1+a_{i3}$	$>$	$2a_{j3}$	

   - Drawing lots versus $D_3^{\underline{\mu}}$:

   $$\begin{aligned} u_3^{max} &= \tfrac{1}{3}\big(\overbrace{1-x_1-y_2}^{a_{13}}+\overbrace{1-x_2-y_1}^{a_{23}}+\overbrace{\ 1\ }^{a_{33}^{max}}\big) \\ &\overset{\mu 1\leq\mu_2}{\leq} 1-\tfrac{2}{3}(x_1+y_1) < 1-\tfrac{1}{2}(x_1+y_1) = 1-\underline{\mu} = a_{33}(D_3^{\underline{\mu}}) \end{aligned} \quad (39)$$

   - Correlated Equilibrium versus $D_3^{\underline{\mu}}$:
     $1-\underline{\mu} > \frac{1}{2}(1-\underline{\mu}+a_{i3})$, $i=1,2$.
   - Simplex minimization for player 3:

   $$\pi_3^{min}(D_3^{\underline{\mu}}) = \tfrac{1}{2} \quad (40)$$

   - Example: $A = \begin{pmatrix} \frac{7}{10} & \frac{3}{10} & 0 \\ \frac{3}{10} & \frac{7}{10} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

2. $D_3^{\overline{\mu}}$

   Constraints for $D_3^{\overline{\mu}}$ to be the best proposal of player 3 (w.l.o.g $\mu_2 = \overline{\mu}$):

1.	$2y_2$	$\geq$	x_2	
2.	$2y_1$	$<$	x_1	
3.	μ_2	$\leq$	$\underline{x}$	
4.	a_{i3}	$\leq$	a_{j3}	$\Big\}\ i,j=1,2\ i\neq j$
5.	$1-\mu_2+a_{i3}$	$>$	$2a_{j3}$	
6.	$\mu_1+2y_2-x_2$	$\geq$	μ_2	
7.	$\mu_1+2y_2-x_2$	$\geq$	μ_2	

---

[10] This minimization is done by Maple. Since the constraints are linear, there exists a finite algorithm to calculate the extremas, which are in the corners of the 4-dimensional hyper-polyeder. Furthermore the number does not exceed 30, so that the problem has less than 100000 corners, which can be checked with a computer within some seconds.

- Drawing lots versus $D_3^{\overline{\mu}}$:

$$\begin{aligned} u_3^{max} &= \tfrac{1}{3}\big(\overbrace{1-x_1-y_2}^{a_{13}}+\overbrace{1-x_2-y_1}^{a_{23}}+\overbrace{1-2y_2-x_2}^{a_{33}^{max}}\big) \\ &= 1-\tfrac{1}{3}(x_1+y_1)-y_2 \end{aligned} \tag{41}$$

  We have

$$\begin{gathered} x_1+y_1 \overset{\mu_2\leq\underline{x}}{\geq} \tfrac{1}{2}(x_2+y_2)) \overset{2y_2\geq x_2}{\geq} \tfrac{3}{2}(x_2-y_2) \\ \Longrightarrow \underbrace{1-\frac{1}{3}(x_1+y_1)-y_2)}_{u_3^{max}} \leq \underbrace{1-\frac{1}{2}(x_2+y_2)}_{a_{33}(D_3^{\overline{\mu}})} \end{gathered} \tag{42}$$

- Correlated Equilibrium versus $D_3^{\overline{\mu}}$:
  Suppose $\pi_3(\text{Correlated equilibrium}) > a_{33}(D_3^{\overline{\mu}}) \Longrightarrow$

$$\overbrace{\frac{1}{2}(1-\mu_1+1-x_1-y_2)}^{\pi_3^{max}(\text{Correlated equilibrium})} > \overbrace{1-\mu_2}^{a_{33}(D_3^{\overline{\mu}})} \Longrightarrow \underbrace{\frac{1}{2}(y_1-x_1)}_{<0} > \underbrace{2-x_2}_{>0} \quad \lightning \tag{43}$$

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\overline{\mu}}) = \frac{3}{7} \tag{44}$$

- Example $A = \begin{pmatrix} \frac{4}{7}-\epsilon & \frac{3}{7}+\epsilon & 0 \\ \frac{2}{7}+\epsilon & \frac{5}{7}-\epsilon & 0 \\ 0 & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$

3. $D_3^{\mu_i}$

   Constraints for $D_3^{\mu_i}$ to be the best proposal of player 3 (w.l.o.g. we consider only the constraints for $D_3^{\mu_1}$):

(i)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1+2y_2-x_2$	$<$	$\underline{x}$
4.	$\mu_1+2y_2-x_2$	$<$	μ_2
5.	a_{i3}	$\leq$	a_{j3}
6.	$1-(\mu_1+2y_2-x_2+\epsilon)+a_{i3}$	$>$	$2a_{j3}$

$\left.\right\}\ i,j=1,2\ i\neq j$

(ii)

1.	$2y_1$	$\geq$	x_1
2.	$2y_2$	$\geq$	x_2
3.	$\mu_1+2y_2-x_2$	$<$	$\underline{x}$
4.	$\mu_1+2y_2-x_2$	$\leq$	$\mu_2+2y_1-x_1$

- Drawing lots versus $D_3^{\mu_1}$:
  For constraint set (i) the same argumentation as in $D_3^{\overline{\mu}}$ holds and for constraint set (ii) we have:

$$\begin{array}{rcl} u_{33}^{max} & = & \frac{1}{3}\big(\overbrace{1-x_1-y_2}^{a_{13}}+\overbrace{1-x_2-y_1}^{a_{23}}+\overbrace{1-2y_1+x_1-\epsilon-2y_2+x_2-\epsilon'}^{a_{33}^{max}}\big) \\ & = & 1-(y_1+y_2)-\frac{1}{3}(\epsilon+\epsilon`) \end{array} \tag{45}$$

and

$$\begin{array}{l} y_1+y_2 \overset{\substack{2y_1\geq x_1 \\ 2y_2\geq x_2}}{\geq} \underline{x} \qquad\qquad \Longrightarrow \\ \underbrace{1-(y_1+y_2)-\frac{1}{3}(\epsilon+\epsilon`)}_{u_3^{max}} < 1-\underline{x} \overset{\mu_1+2y_2-x_2<\underline{x}}{\leq} \underbrace{1-(\mu_1+2y_2-x_2+\epsilon)}_{a_{33}(D_3^{\mu_1})} \\ \qquad\qquad\qquad\qquad (\epsilon \leq \underline{x}-(\mu_1+2y_2-x_))) \end{array} \tag{46}$$

- Correlated Equilibrium versus $D_3^{\mu_i}$
  (a) For constraint set (i) we have $\mu_2 > \mu_1+2y_2-x_2$
  (b) For constraint set (ii) we have $\mu_2+2y_1-x_1 \geq \mu_1+2y_2-x_2$
- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\mu_i}) > \frac{3}{8} \tag{47}$$

- Examples ($\epsilon' < \epsilon$): $A^{(i)} = \begin{pmatrix} \frac{4}{7} & \frac{3}{7}-\frac{\epsilon}{2} & \frac{\epsilon}{2} \\ \frac{2}{7} & \frac{7}{10} & 2\epsilon \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \end{pmatrix}$ $A^{(ii)} = \begin{pmatrix} \frac{5}{8}+2\epsilon & \frac{3}{8}-4\epsilon & 2\epsilon \\ \frac{3}{7} & \frac{5}{8} & 0 \\ \frac{4}{8}+\epsilon & \frac{1}{8}-8\epsilon+\epsilon' & \frac{3}{8}+7\epsilon-\epsilon' \end{pmatrix}$

4. $D_3^{\underline{x}}$

Constraints for $D_3^{\underline{x}}$ to be the best proposal of player 3:

(i)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	μ_2	$\geq$	$\underline{x}$
4.	$\mu_1+2y_2-x_2$	$>$	$\underline{x}$

- Drawing lots versus $D_3^{\underline{x}}$:

$$u_3^{max} = \frac{1}{3}(a_{13}+a_{23}+\underbrace{1-(\mu_1+2y_2-x_2+\epsilon)}_{a_{33}(D_3^{\underline{x}})}) \overset{\substack{a_{13}\leq 1-\underline{x} \\ a_{13}\leq 1-\underline{x}}}{\leq} a_{33}(D_3^{\underline{x}})$$

- Correlated Equilibrium versus $D_3^{\underline{x}}$:

  For a correlated Equilibrium, we need $\mu_1 = \underline{\mu}$. Otherwise it could not be better than $D_3^{\underline{x}}$, since $1 - \mu_1$ is part of the correlated payoff of player 3 and $1 - \mu_1 \leq 1 - \underline{x}$.

  Additionally we need the following Rank-Matrix:

$$\begin{pmatrix} x_1 & y_2 & 1 - x_1 - y_2 \\ y_1 & x_2 & 1 - y_1 - x_2 \\ \mu_1 & 0 & 1 - \mu_1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2^* & 2^* \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix} \tag{48}$$

  This implies the following three conditions, for the existence of a correlated equilibrium:

  (a) $\mu_1 < \mu_2$

  (b) $a_{23} < a_{13}$

  (c) $1 - \mu_1 + a_{23} < 2a_{13}$

  Two cases are possible $x_1 = \underline{x} \vee x_2 = \underline{x}$

  (a) Suppose $x_1 = \underline{x} \Longrightarrow$

$$\begin{gathered} 1 - \mu_1 + a_{23} < 2a_{13} \Longleftrightarrow 3x_1 + 4y_2 < 3y_1 + 2x_2 \quad \text{and} \\ 2y_1 < x_1 \Longleftrightarrow 3x_1 + 4y_2 > 6y_1 + 4y_2 \overset{2y_2 \geq x_2}{\geq} 6y_1 + 2x_2 \geq 3y_1 + 2x_2 \quad \lightning \end{gathered} \tag{49}$$

  (b) Suppose $x_2 = \underline{x} \Longrightarrow \rho$ is not symmetric anymore

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\underline{x}}(i)) > \frac{3}{7} \tag{50}$$

- Example: $A^{(i)} = \begin{pmatrix} \frac{4}{7} - \epsilon & \frac{3}{7} + \epsilon & 0 \\ \frac{2}{7} - 11\epsilon & \frac{5}{7} - 3\epsilon & 14\epsilon \\ \frac{4}{7} - \epsilon & 0 & \frac{3}{7} + \epsilon \end{pmatrix}$

(ii)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	μ_2	$\leq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\geq$	μ_2
5.	a_{13}	$<$	a_{23}
6.	$1 - \mu_2 + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$\leq$	$1 - \underline{x}$

($a_{13} > a_{23}$ not possible)

Suppose $a_{13} > a_{23} \Longrightarrow$ condition 6. converts to $1 - \mu_2 + a_{23} \leq 2a_{13} \Longrightarrow$

$$4x_1 - y_1 + 3y_2 \leq 3x_2 \overset{\mu_2 \leq \mu_1 + 2y_2 - x_2}{\leq} x_1 + y_1 + 3y_2 \Longrightarrow 3x_1 \leq 2y_1 \quad \lightning \tag{51}$$

- Drawing lots versus $D_3^{\underline{x}}$:
  If $(x_1 = \underline{x})$ constraint (4.) fails and if $(x_2 = \underline{x})$ the additionally needed constraint $\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23}) > 1 - \underline{x}$ fails.
- Correlated equilibrium versus $D_3^{\underline{x}}$:
  For $D_3 = (a_{31}, 0, 1 - a_{31})$ we need for a correlated equilibrium to be possible

$$\begin{pmatrix} x_1 & y_2 & 1 - x_1 - y_2 \\ y_1 & x_2 & 1 - y_1 - x_2 \\ a_{31} & 0 & 1 - a_{31} \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2^* & 2^* \\ 1 & 3 & 1 \\ 2^* & 1 & 3 \end{pmatrix} \tag{52}$$

  and $a_{23} < a_{13}$, which implies $1 - a_{31} + a_{23} \leq 2a_{13}$.

  Suppose $1 - a_{31} + a_{23} \leq 2a_{13} \Longrightarrow a_{31} \geq \underbrace{2x_1 - y_1}_{> x_1} + \underbrace{2y_2 - x_2}_{\geq 0} \lightning$

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\underline{x}}(ii)) = \frac{3}{7} \tag{53}$$

- Example: $A^{(ii)} = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} & 0 \\ \frac{1}{14} & \frac{4}{7} & \frac{5}{14} \\ \frac{4}{7} & 0 & \frac{3}{7} \end{pmatrix}$

(iii)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$<$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$\leq$	$\underline{x}$
4.	$\mu_1 + 2y_2 - x_2$	$\leq$	μ_2
5.	a_{13}	$<$	a_{23}
6.	$1 - (\mu_1 + 2y_2 - x_2) + a_{13}$	$\leq$	$2a_{23}$
7.	$\frac{1}{2}(1 - \mu_2 + a_{23})$	$\leq$	$1 - \underline{x}$
8.	$1 - (2y_2 - x_2) + a_{13}$	$>$	$2a_{23}$
9.	$\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23})$	$\leq$	$1 - \underline{x}$

($a_{23} < a_{13}$ is not possible.)

Suppose $a_{23} < a_{13} \Longrightarrow$ condition 6. converts to $1 - (\mu_1 + 2y_2 - x_2) + a_{23} \leq 2a_{13}$ $\Longrightarrow x_1 < y_1$.

(iv)

1.	$2y_2$	$\geq$	x_2
2.	$2y_1$	$\geq$	x_1
3.	$\mu_1 + 2y_2 - x_2$	$\geq$	$\underline{x}$
4.	$\mu_2 + 2y_1 - x_1$	$\geq$	$\underline{x}$

(v)

1.	$2y_2 < x_2$
2.	$2y_1 < x_1$
3.	$a_{i3} \leq a_{j3}$
4.	$1 - \underline{\mu} + a_{i3} \leq 2a_{j3}$
5.	$\frac{1}{2}(1 - \mu_i + a_{j3}) \leq 1 - \underline{x}$
6.	$\frac{1}{3}(1 + a_{13} + a_{23}) \leq 1 - \underline{x}$

$\left.\right\}$ $i, j = 1, 2 \; i \neq j$

- Drawing lots versus $D_3^{\underline{x}}$:

  For the constraint set (iii) the same argumentation as for constraint set (i) of $D_3^{\mu_i}$ holds. For the constraint set (iv) $u_3^{max} \leq 1 - \underline{x}$ follows directly from constraints 3. and 4..

- Correlated equilibrium versus $D_3^{\underline{x}}$:

  For the constraint set (iii) the same argumentation holds as for constraint sets (i) and (ii) of $D_3^{\underline{x}}$. For the constraint set (iv) π_3^{max}(Correlated equilibrium) $\leq 1 - \underline{x}$ follows directly from constraints 3. and 4..

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\underline{x}}(\text{iii})) = \frac{3}{8} \quad \pi_3^{min}(D_3^{\underline{x}}(\text{iv})) = \frac{3}{8} \quad \pi_3^{min}(D_3^{\underline{x}}(\text{v}) > \frac{3}{7} \tag{54}$$

- Examples: $A^{(iv)} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \\ \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}$ $A^{(v)} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} & 0 \\ \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}$

$$A^{(v)} = \begin{pmatrix} \frac{4}{7}-3\epsilon & \frac{1}{7}+\epsilon & \frac{2}{7}+\epsilon \\ \frac{2}{7}-5\epsilon & \frac{5}{7}+5\epsilon & \frac{8}{21} \\ \frac{4}{7}-3\epsilon & 0 & \frac{3}{7}+3\epsilon \end{pmatrix}$$

5. D_3^{Cor}

   (a) Constraints as in $D_3^{\underline{x}}$ set (ii) and constraint (7.) reversed:
   7'. $\frac{1}{2}(1 - \mu_2 + a_{23}) > 1 - \underline{x}$

   (b) Constraints as in $D_3^{\underline{x}}$ set (iii) and constraint (7.) reversed:
   7'. $\frac{1}{2}(1 - (\mu_2 + a_{23}) > 1 - \underline{x}$ and constraint (9.) replaced by
   9'. $\frac{1}{3}(1 - (2y_2 - x_2) + a_{13} + a_{23}) \leq \frac{1}{2}(1 - \mu_2 + a_{23})$

   (c) Constraints as in $D_3^{\underline{x}}$ set (vi) and constraint (5.) reversed:
   5'.$\frac{1}{2}(1 - \mu_i + a_{j3}) > 1 - \underline{x}$
   and constraint (6.) replaced by
   6'. $\frac{1}{3}(1 + a_{13} + a_{23}) \leq \frac{1}{2}(1 - \mu_i + a_{j3})$

(That (a) is better than drawing lots for player 3 follows from the same argumentation as in constraint set (ii) of $D_3^{\underline{x}}$)

6. D_3^L

   (a) Constraints as in $D_3^{\underline{x}}$ set (iii) and constraint (9.) reversed:
   9'. $\frac{1}{3}(1-(2y_2-x_2)+a_{13}+a_{23}) > 1-\underline{x}$
   and constraint (7.) replaced by
   7'. $\frac{1}{3}(1-(2y_2-x_2)+a_{13}+a_{23}) > \frac{1}{2}(1-\mu_2+a_{23})$

   (b) Constraints as in $D_3^{\underline{x}}$ set (vi) and constraint (6.) reversed:
   6'.$\frac{1}{3}(1+a_{13}+a_{23}) > 1-\underline{x}$
   and constraint (5.) replaced by
   5'. $\frac{1}{3}(1+a_{13}+a_{23}) > \frac{1}{2}(1-\mu_i+a_{j3})$

Since the minimum payoff of $D_3^{\underline{x}}$ is given by $\pi_3^{min}(D_3^{\underline{x}}) = \frac{3}{8}$ and the constraints for D_3^{cor} and D_3^L directly imply that $\pi_3(D_3^{cor}) > 1-\underline{x}$ and $\pi_3(D_3^L) > 1-\underline{x}$ we obtain $\pi_3^{min}(D_3^{cor}) > \frac{3}{8}$ $\pi_3^{min}(D_3^L) > \frac{3}{8}$.

Altogether we obtain

$$\min_{\alpha\in\{D_3^{\underline{\mu}},D_3^{\overline{\mu}},D_3^{\mu_i},D_3^{\underline{x}},D_3^{Cor},D_3^L\}}\{\pi_3(\alpha)\} = \min_{\rho \text{ symmetric}}\{\pi_3\} = \frac{3}{8} \tag{55}$$

■

7.1.2 Proof of Corollary 1

Given $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the constraints of $D_3^{\underline{\mu}}$ hold with $\mu_1 = \mu_2 = \underline{\mu} = \frac{1}{2}$, which implies that $D_3 = (\frac{1}{2}, 0, \frac{1}{2})$ or $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ are the best reaction of player 3 resulting in the payoffs $\pi_1 = \pi_2 = \frac{1}{4}$ and $\pi_3 = \frac{1}{2}$

■

We also obtain the following corollary, which will be used later on:

Corollary 5
Given symmetric $\rho \Longrightarrow \pi_2 > \underline{x}$ iff D_3^{cor} or D_3^L is the best reaction of player 3.

7.1.3 Proof of Proposition 2

There exist six possibilities for ρ to be non symmetric:

$$\begin{array}{cccc} & & \rho_1^{ns} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} & \rho_{1'} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \\ \rho_3^{ns} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} & \rho_{3'}^{ns} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} & \rho_{3''}^{ns} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} & \rho_{3'''}^{ns} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \end{array} \tag{56}$$

In the following we treat only ρ_1^{ns}, since any other case arises by interchanging rows and observing that for ρ_3^{ns} nothing changes qualitatively. For ρ_1^{ns} A is given by:

$$A = \begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{aligned} a_{13} &= 1 - y_1 - y_2 \\ a_{23} &= 1 - x_1 - x_2 \\ a_{33} &= 1 - a_{31} - a_{32} \end{aligned} \tag{57}$$

Firstly, we observe, that player 3 has to offer at least player i more than x_i to gain more than $\pi_3 = 1 - x_1 - x_2$, because otherwise D_2 contains two maximum shares and is a single proposal equilibrium.[11]

As in the proof of proposition 1 we look for the best offer D_3 and see that the following properties of a are relevant:

- $1 \geq a_{3i} > 2x_i - y_i \Longrightarrow \Phi(x_i) = 2, \Phi(y_i) = 1, \Phi(a_{3i}) = 3$
- $y_i > a_{3i} > 2y_i - x_i \Longrightarrow \Phi(x_i) = 3, \Phi(y_i) = 2, \Phi(a_{3i}) = 1$

With these properties, we can distinguish four different kinds of proposals D_3, which will be discussed in detail below:

$$D_3^{\underline{x}^\epsilon} \qquad D_3^o \qquad D_3^3 \qquad D_3^{L^\epsilon}$$

1. $D_3^{\underline{x}^\epsilon}$

   Player 3 offers player 1 or 2 $\underline{x} + \epsilon$ and the rest to himself, such that D_3 is a single proposal equilibrium and he receives a payoff $\pi_3 = 1 - \underline{x} - \epsilon$ (i.e):

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ \underline{x} + \epsilon & 0 & 1 - \underline{x} - \epsilon \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 2^* & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix} \vee \begin{pmatrix} 1 & 2 & 3 \\ 2^* & 3 & 1 \\ 3 & 1 & 2^* \end{pmatrix} \tag{58}$$

2. D_3^o

   Player 3 offers player i $2x_i - y_i + \epsilon$ and the rest to himself, such that D_1 is a single proposal equilibrium and he receives a payoff $\pi_3 = 1 - y_1 - y_2$ (i.e):

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2^* & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix} \tag{59}$$

3. D_3^3

   Player 3 offers player i $2x_i - y_i + \epsilon$ ($i \in \{1, 2\}$)and the rest to himself, such that D_3 is a single proposal equilibrium and he receives a payoff $\pi_3 = 1 - (2x_i - y_i + \epsilon)$ (i.e):

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2^* \end{pmatrix} \tag{60}$$

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[11] If player 3 offers player 1 only x_1 D_2 is still the only proposal with two maxima or D_2 and D_3 are both voting equilibria, but not correlated.

4. $D_3^{L^\epsilon}$

   Player 3 offers player i $2x_i - y_i + \epsilon$ and the rest to himself, such that the voting stage leads to a tie-break decision and he receives a payoff payoff $\pi_3 = \frac{1}{3}(1 - (2x_1 - y_1 + \epsilon) + a_{13} + a_{23})$ (i.e):

$$\begin{pmatrix} y_1 & y_2 & a_{13} \\ x_1 & x_2 & a_{23} \\ 2x_1 - y_1 + \epsilon & 0 & 1 - (2x_1 - y_1 + \epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \tag{61}$$

1. $D_3^{\underline{x}^\epsilon}$

   Constraints for $D_3^{\underline{x}^\epsilon}$ to be the best proposal of player 3:

(ii)

1.	$y_1 + y_2$	$>$	$\underline{x}$
2.	$2y_1$	$<$	x_1
3.	$2y_2$	$<$	x_2

Since player 3 needs to offer at least player i $a_{3i} = \underline{x} + \epsilon$ to prevent D_2 to be a single proposal equilibrium and $1 - \underline{x} + \epsilon > a_{i3}$ $i = 1, 2$, $D_3^{\underline{x}^\epsilon}$ is better than all other proposals.

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^{\underline{x}^\epsilon}) > \tfrac{1}{2} \tag{62}$$

- Example:

$$A^{(i)} = \begin{pmatrix} \frac{1}{4} - 3\epsilon & \frac{1}{4} + \epsilon & \frac{1}{2} + 2\epsilon \\ \frac{1}{2} - 4\epsilon & \frac{1}{2} + 4\epsilon & 0 \\ \frac{1}{2} - 4\epsilon + \epsilon' & 0 & \frac{1}{2} + 4\epsilon - \epsilon' \end{pmatrix} \quad \epsilon' < \epsilon$$

2. D_3^o

   Constraints for D_3^o to be the best proposal of player 3:

1.	$y_1 + y_2$	$\leq$	$\underline{x}$
2.	$2y_i$	$\geq$	x_i
3.	$2x_j - y_j$	$<$	1

$\left.\right\}$ $i, j = 1, 2 \; i \neq j$

Since $\pi_3 = 1 - y_1 - y_2$ is only ϵ worse than the payoff of $D_3^{\underline{x}^\epsilon}$, D_3^o is the best offer.

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^o) = \tfrac{1}{2} \tag{63}$$

- Example:

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3. D_3^3

Constraints for D_3^o to be the best proposal of player 3:

(i)

1.	$y_1 + y_2 \leq \underline{x}$
2.	$2y_1 < x_1$
3.	$2y_2 < x_2$
4.	$2x_i - y_i < 1$
5.	$1 - (2x_i - y_i + \epsilon) \geq \frac{1}{2}(a_{13} + a_{23})$[12]

$\Big\}\ (i = 1, 2)$

(ii)

1.	$y_1 + y_2 \leq \underline{x}$
2.	$2y_i < x_i$
3.	$2y_j \geq x_j$
5.	$2x_j - y_j < 1$
6.	$2x_i - y_i \geq 1$
7.	$1 - (2x_j - y_j + \epsilon) \geq \frac{1}{2}(a_{13} + a_{23})$

$\Big\}\ i, j = 1, 2\ i \neq j$

Since in a tie-break decision player 3 cannot gain more than $\frac{1}{2}(a_{13} + a_{23}) - \epsilon'$ and $D_3^{x^\epsilon}$ and D_3^o are not an option, D_3^3 is the best proposal.

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^3) > \frac{1}{4} \tag{64}$$

- Examples ($\epsilon' < \frac{\epsilon}{2}$):

$$A^{(i)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}-\epsilon & \frac{1}{2}+\epsilon \\ \frac{1}{2}+\epsilon & \frac{1}{2}-\epsilon & 0 \\ 0 & \frac{3}{4}-\epsilon+\epsilon' & \frac{1}{4}+\epsilon-\epsilon' \end{pmatrix} \quad A^{(ii)} = \begin{pmatrix} 0 & \frac{1}{3}+\epsilon & \frac{2}{3}-\epsilon \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3}-\epsilon+\epsilon' & \frac{1}{3}+\epsilon-\epsilon' \end{pmatrix}$$

4. $D_3^{L^\epsilon}$

Constraints for $D_3^{L^\epsilon}$ to be the best proposal of player 3:

(i)

1.	$y_1 + y_2 \leq \underline{x}$
2.	$2y_1 < x_1$
3.	$2y_2 < x_2$
4.	$2x_i - y_i < 1$
5.	$1 - (2x_i - y_i + \epsilon) < \frac{1}{2}(a_{13} + a_{23})$
6.	$1 - (2x_j - y_j + \epsilon) < \frac{1}{2}(a_{13} + a_{23})$
7.	$1 - (2x_j - y_j) \leq 1 - (2x_i - y_i)$

$\Bigg\}\ i, j = 1, 2\ i \neq j$

---

[12] If $2x_i - y_i < 1$ and $1-(2x_i - y_i + \epsilon)$ hold simultaneously for $i = 1, 2$, player 3 chooses $\min_{i=1,2}\{2x_i - y_i\}$

(ii) $i, j = 1, 2\ i \neq j$

1.	$y_1 + y_2$	$\leq$	$\underline{x}$	
2.	$2y_i$	$<$	x_i	$\left.\right\}$ $i, j = 1, 2\ i \neq j$
3.	$2y_j$	$\geq$	x_j	
4.	$2x_j - y_j$	$<$	1	
5.	$2x_i - y_i$	$\geq$	1	
6.	$1 - (2x_j - y_j + \epsilon)$	$<$	$\frac{1}{2}(a_{13} + a_{23})$	

Since all other kinds of proposals are excluded $D_3^{L^\epsilon}$ is the best proposal.

- Simplex minimization for player 3:

$$\pi_3^{min}(D_3^3) > \tfrac{1}{4} \tag{65}$$

- Examples ($\epsilon > \epsilon'$):

$$A^{(i)} = \begin{pmatrix} \frac{1}{4}-\epsilon & \frac{1}{4}-\epsilon & \frac{1}{2}+2\epsilon \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{4}-\epsilon+\epsilon' & \frac{1}{4}+\epsilon-\epsilon' \end{pmatrix} \quad A^{(ii)} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3}+\epsilon & \frac{1}{3}-\epsilon \end{pmatrix}$$

If we calculate the feasible set of $(D_3^{\underline{x}^\epsilon} \cup D_3^o \cup D_3^3 \cup D_3^{L^\epsilon})$ we obtain

1.	$y_1 + y_2$	$\leq$	$\underline{x}$
2.	$2y_1$	$<$	x_2
3.	$2y_2$	$<$	x_2
5.	$2x_1 - y_1$	$\geq$	1
6.	$2x_2 - y_2$	$\geq$	1

is missing. The only combination of $(D_1\,, D_2)$ to fulfill this is given by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This is also the submatrix a, which is excluded in proposition 2.

Altogether we obtain for $a \neq \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \vee a \neq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$

$$\min_{\alpha \in \{D_3^{\underline{x}^\epsilon}, D_3^o, D_3^3, D_3^{L^\epsilon}\}} = \min_{\rho \text{ non-symmetric}} \{\pi_3\} > \frac{1}{4} \tag{66}$$

■

7.1.4 Proof of Corollary 2

W.o.l.g we assume $a = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Longrightarrow$ player 3 must offer player i at least $a_{3i} = \mu_i$ and player j at least $a_{3j} = x_j + \epsilon$ $(i, j = 1, 2\ i \neq j)$ in order to avoid that D_2 is a single proposal equilibrium with $\pi_3 = 0$. $\Longrightarrow D_3 = (\frac{1}{4}, \frac{1}{2} + \epsilon, \frac{1}{8} - \epsilon)$ or $D_3 = (\frac{1}{2} + \epsilon, \frac{1}{4}, \frac{1}{8} - \epsilon)$ with the correlated equilibrium C_{23} is the best reaction of player 3 resulting in the payoffs $\pi_1 = \pi_2 = \frac{7}{16} + \frac{\epsilon}{4}$ and $\pi_3 = \frac{1}{8} - \frac{\epsilon}{2}$.

■

7.2 Proof of propositions 3, 4 and 5 and of corollaries 3 and 4

We proof these propositions and corollaries by constructing D_2 such, that we satisfy different constraint sets of the proofs of propositions 1 and 2. But since we have shown propositions 1 and 2 only for representative matrices a, sometimes we have to interchange indices in the constraint sets, in order to adapt them for the following proofs.

7.2.1 Proof of Proposition 3

1. Suppose that $D_1 \in \mathcal{A} \Longrightarrow$ player 2's payoff will be at least $\pi_2 > \frac{3}{8}$ and $\rho = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, if he proposes $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ or $D_2 = (a_{12}, a_{11} - \epsilon, a_{13} + \epsilon)$.

$\mathcal{A}_1$) Suppose $D_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $D_3 = (0, \frac{1}{2}, \frac{1}{2})$ is a single proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2^* & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{67}$$

This can be satisfied if the constraint set (i) of $D_3^{\underline{x}}$ holds. The feasible set $\mathcal{S}_{\mathcal{A}_1}$ is then determined by

$$\begin{array}{lrcccrcl} 1. & 2a_{21} & \geq & a_{11} & \Longrightarrow & a_{11} & \leq & 1 \\ 2. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{12} & < & \frac{1}{4} \\ 3. & \mu_1 & \geq & \underline{x} & \Longrightarrow & a_{11} & \geq & \frac{1}{2} \\ 4. & \mu_2 + 2y_1 - x_1 & \geq & \underline{x} & \Longrightarrow & a_{12} & \geq & 2a_{11} - \frac{3}{2} \end{array}$$

After comparison of the inequalities, only 4. is remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_1} \quad = \quad \left\{ \left(\tfrac{1}{2}, \tfrac{3}{4}\right] \times \left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, \tfrac{5}{6}\right] \times \left[2a_{11} - \tfrac{3}{2}, 1 - a_{11}\right] \right\} \tag{68}$$

(See figure 1)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{A}_1) = \frac{1}{2} \tag{69}$$

$\mathcal{A}_2$) See $\mathcal{A}_1$, but now with constraint set (v) of $D_3^{\underline{x}}$. This requires

$$\begin{array}{lrclcrcl}
1. & 2a_{21} & \geq & a_{11} & \Longrightarrow & a_{11} & \leq & 1 \\
2. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} & \geq & \frac{1}{4} \\
3. & \mu_2 + 2a_{21} - a_{11} & \geq & \underline{x} & \Longrightarrow & a_{12} & \geq & 2a_{11} - \frac{3}{2} \\
4. & \mu_1 + 2a_{12} - a_{22} & \geq & \underline{x} & \Longrightarrow & a_{12} & \geq & \frac{3}{8} - \frac{1}{4}a_{11})
\end{array}$$

After comparison of the inequalities, only 2. is remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_2} \quad = \quad \left\{ \left(\tfrac{1}{2}, \tfrac{3}{4} \right] \times \left[\tfrac{1}{4}, 1 - a_{11} \right] \right\} \tag{70}$$

(See figure 2)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{A}_2) = \frac{1}{2} \tag{71}$$

$\mathcal{A}_3$) Suppose $D_2 = (a_{12}, a_{11} - \epsilon, a_{13} + \epsilon)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is a single proposal equilibrium $\Longrightarrow$ This requires the following rank matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{11} - \epsilon & a_{13} + \epsilon \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2^* & 3 \end{pmatrix} \tag{72}$$

This can be satisfied if the constraint set of $D_3^{\underline{\mu}}$ holds (note that $\mu_2 = \underline{\mu}$). The feasible set $\mathcal{S}_{\mathcal{A}_2}$ is then determined by

$$\begin{array}{lrclcrcl}
1. & 2a_{21} & < & a_{11} & \Longrightarrow & a_{11} & > & \frac{3}{4} \\
2. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{11} & > & \frac{3}{4} \\
3. & a_{23} & \geq & a_{13} & \Longrightarrow & \epsilon & > & 0 \\
4. & 1 - \mu_2 + a_{13} & > & 2a_{23} & \Longrightarrow & a_{12} & \geq & \epsilon' - a_{11}
\end{array}$$

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{A}_3} \quad = \quad \left\{ \left(\tfrac{3}{4}, 1 \right] \times [0, 1 - a_{11}] \right\} \tag{73}$$

(See figure 3)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{A}_3) > \frac{3}{8} \tag{74}$$

2. Suppose $D_1 \in \mathcal{B} \implies$ player 2's payoff will be at least $\pi_2 > \frac{3}{8}$ and $\rho = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, if he proposes $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$, $D_2 = (\frac{3}{4} - a_{11} + 3\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, a_{11} + a_{12} - \frac{1}{2} - 5\epsilon)$, $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$, $D_2 = (a_{12} - \frac{1}{4} - 2\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, 0)$, $D_2 = (\frac{5}{8} + 4\epsilon, \frac{3}{8} - 4\epsilon, 0)$, $D_2 = (a_{12} + 4\epsilon, 1 - (a_{12} + 4\epsilon), 0)$ or $D_2 = (1, 0, 0)$.

$\mathcal{B}_1$) Suppose $D_2 = (a_{12} + \epsilon, 1 - (a_{12} + \epsilon), 0)$ and $D_3 = (0, a_{12}, 1 - a_{12})$ is a single proposal equilibrium $\implies$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + \epsilon & 1 - (a_{12} + \epsilon) & 0 \\ 0 & a_{12} & 1 - a_{12} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{75}$$

This can be satisfied if the constraint set (ii) of $D_3^{\underline{x}}$ holds (by construction we have $\underline{x} = a_{12}$). The feasible set $\mathcal{S}_{\mathcal{B}_1}$ is then determined by

$$\begin{array}{lrcllcl}
1. & 2a_{22} & \geq & a_{12} & \implies & a_{12} & < & \frac{2}{3} \\
2. & 2a_{11} & < & a_{21} & \implies & a_{12} & \geq & 2a_{11} \\
3. & \mu_2 & < & \underline{x} & \implies & a_{12} & \geq & \frac{1}{2} \\
4. & \mu_1 + 2a_{22} - a_{12} & \geq & \mu_2 & \implies & a_{12} & < & \frac{3}{5} + \frac{a_{11}}{5} \\
5. & a_{23} & < & a_{13} & \implies & a_{12} & < & 1 - a_{11} \\
6. & 1 - \mu_2 + a_{23} & \leq & 2a_{13} & \implies & a_{12} & < & \frac{3}{4} - a_{11} \\
7. & \frac{1}{2}(1 - \mu_2 + a_{13}) & \leq & 1 - \underline{x} & \implies & a_{12} & \leq & \frac{1}{2} + a_{11}
\end{array}$$

After comparison of the inequalities, only 3., 6., and 7. are remaining.

$$\implies \quad \mathcal{S}_{\mathcal{B}_1} \quad = \quad \left\{ \left[0, \tfrac{1}{8}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + a_{11}\right] \cup \left[\tfrac{1}{8}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4} - a_{11}\right) \right\} \tag{76}$$

(See figure 5)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_1) = \frac{1}{2} \tag{77}$$

$\mathcal{B}_2$) Suppose $D_2 = (\frac{3}{4} - a_{11} + 3\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, a_{11} + a_{12} - \frac{1}{2} - 5\epsilon)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is single proposal equilibrium $\implies$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{3}{4} - a_{11} + 3\epsilon & \frac{3}{4} - a_{12} + 2\epsilon & 1 - a_{21} - a_{22} \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{78}$$

This can be satisfied if the constraints of D_3^{μ} hold. The feasible set $\mathcal{S}_{\mathcal{B}_2}$ is then determined by

$$\begin{array}{lrcllcl}
1. & \mu_2 & < & \mu_1 & \implies & \epsilon & > & 0 \\
2. & 2a_{11} & < & a_{21} & \implies & a_{11} & \leq & \frac{1}{4} \\
3. & a_{23} & \leq & a_{13} & \implies & a_{12} & \leq & \frac{3}{4} - a_{11} \\
4. & 1 - \mu_2 + a_{23} & > & 2a_{13} & \implies & a_{12} & > & \frac{5}{8} - a_{11}
\end{array}$$

After comparison of the inequalities, only 3. and 4. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_2} \quad = \quad \left\{ \left[0, \tfrac{1}{8}\right] \times \left(\tfrac{5}{8} - a_{11}, \tfrac{3}{4} - a_{11}\right] \cup \left(\tfrac{1}{8}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4} - a_{11}\right] \right\} \tag{79}$$

(See figure 6)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_2) > \frac{3}{8} \tag{80}$$

$\mathcal{B}_3$) Suppose $D_2 = (a_{12}+\epsilon, 1-(a_{12}+\epsilon), 0)$ and $D_3 = (0, a_{12}, 1-a_{12})$ is single proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}+\epsilon & 1-(a_{12}+\epsilon) & 0 \\ 0 & a_{12} & 1-a_{12} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} \tag{81}$$

This can be satisfied if the constraint set (iv) of $D_3^{\frac{x}{3}}$ holds. The feasible set $\mathcal{S}_{\mathcal{B}_3}$ is then determined by

$$\begin{array}{lrcccll} 1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < \; 2a_{11} \\ 2. & 2a_{11} & \geq & a_{21} & \Longrightarrow & a_{12} & < \; \tfrac{2}{3} \\ 3. & \mu_1 + 2a_{22} - a_{12} & \geq & a_{12} & \Longrightarrow & a_{12} & < \; \tfrac{1}{7}a_{11} + \tfrac{4}{7} \\ 4. & \mu_2 + 2a_{11} - a_{21} & \geq & a_{12} & \Longrightarrow & a_{12} & < \; \tfrac{1}{4} + a_{11} \end{array}$$

After comparison of the inequalities only 4. is remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_3} \quad = \quad \left\{ \left[\tfrac{1}{4}, \tfrac{3}{8}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{4} + a_{11}\right) \cup \left(\tfrac{3}{8}, \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, 1 - a_{11}\right] \right\} \tag{82}$$

(See figure 7)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_3) = \frac{1}{2} \tag{83}$$

$\mathcal{B}_4$) Suppose $D_2 = (a_{12} - \frac{1}{4} - 2\epsilon, \frac{3}{4} - a_{12} + 2\epsilon, 0)$ and $D_3 = (0, \mu_2, 1-\mu_2)$ is a single proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} - \frac{1}{4} - 2\epsilon & \frac{3}{4} - a_{12} + 2\epsilon & 0 \\ 0 & \mu_2 & 1-\mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{84}$$

This can be satisfied if the constraints of D_3^{μ} hold (and a_{22} has to be non negative). The feasible set $\mathcal{S}_{\mathcal{B}_4}$ is then determined by

$$\begin{array}{lrcccll} 1. & \mu_2 & < & \mu_1 & \Longrightarrow & a_{12} & > \; \tfrac{1}{2} - a_{11} \\ 2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & > \; 2a_{11} - \tfrac{1}{4} \\ 3. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq \; 1 - a_{11} \\ 4. & 1 - \mu_2 + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & > \; \tfrac{11}{16} - a_{11} \\ 5. & a_{22} & \geq & 0 & \Longrightarrow & a_{12} & \leq \; \tfrac{3}{4} \end{array}$$

After comparison of the inequalities, only 2., 4. and 5. are remaining.

$$\begin{aligned} \Longrightarrow \quad \mathcal{S}_{\mathcal{B}_4} \quad = \quad & \left\{\left[0, \tfrac{3}{16}\right] \times \left(\tfrac{11}{16} - a_{11}, \tfrac{3}{4}\right] \cup \left(\tfrac{3}{16}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4}\right] \cup \right. \\ & \left. \left(\tfrac{1}{4}, \tfrac{3}{8}\right] \times \left[\tfrac{1}{2}, 1 - a_{11}\right] \cup \left[\tfrac{3}{8}, \tfrac{5}{12}\right] \times \left(2a_{11} - \tfrac{1}{4}, 1 - a_{11}\right]\right\} \end{aligned} \tag{85}$$

(See figure 8)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_4) > \frac{3}{8} \tag{86}$$

$\mathcal{B}_5$) Suppose $D_2 = (\frac{5}{8} + 4\epsilon, \frac{3}{8} - 4\epsilon, 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ implies the correlated equilibrium $C_{13} \Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{5}{8} + 4\epsilon & \frac{3}{8} - 4\epsilon & 0 \\ 0 & \mu_2 & 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2^* \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{87}$$

This can be satisfied if the constraint set (b) of D_3^{cor} holds ($a_{12} = \underline{x}$). The feasible set $\mathcal{S}_{\mathcal{B}_5}$ is then determined by and payoff greater than $\frac{3}{8}$

$$\begin{array}{lrcllcl}
1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < \frac{3}{4} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & < \frac{3}{16} \\
3. & \mu_1 + 2a_{22} - a_{12} & \leq & \mu_2 & \Longrightarrow & a_{12} & \geq \frac{7}{12} + \frac{1}{3}a_{11} \\
4. & \mu_1 + 2a_{22} - a_{12} & \leq & a_{12} & \Longrightarrow & a_{12} & \geq \frac{17}{32} + \frac{1}{4}a_{11} \\
5. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq 1 - a_{11} \\
6. & 1 - (\mu_1 + 2a_{22} - a_{12}) + a_{23} & \leq & 2a_{13} & \Longrightarrow & a_{12} & < \frac{11}{16} - \frac{1}{2}a_{11} \\
7. & \frac{1}{2}(1 - \mu_2 + a_{13}) & \leq & 1 - a_{12} & \Longrightarrow & a_{12} & \geq \frac{3}{8} + 2a_{11} \\
8. & 1 - (2a_{22} - a_{12}) + a_{23} & > & 2a_{13} & \Longrightarrow & a_{12} & \geq \frac{7}{12} - \frac{2}{3}a_{11} \\
9. & \frac{1}{3}(1 - (2a_{22} - a_{12}) + a_{23} + a_{13}) & < & \frac{1}{2}(1 - \mu_2 + a_{13}) & \Longrightarrow & a_{12} & \leq \frac{47}{72} - \frac{2}{9}a_{11} \\
10. & a_{12} & < & a_{21} & \Longrightarrow & a_{12} & \leq \frac{5}{8}
\end{array}$$

After comparison of the inequalities only 3. and 10. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_5} \quad = \quad \left\{\left[0, \tfrac{1}{6}\right] \times \left(\tfrac{7}{12} + \tfrac{1}{3}a_{11}, \tfrac{5}{8}\right)\right\} \tag{88}$$

(See figure 9)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_5) > \frac{51}{96} \tag{89}$$

$\mathcal{B}_6$) Suppose $D_2 = (a_{12} + 4\epsilon, 1 - (a_{12} + 4\epsilon), 0)$ and $D_3 = (0, a_{12}, 1 - a_{12})$ implies the correlated equilibrium $C_{13} \Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} + 4\epsilon & 1 - (a_{12} + 4\epsilon) & 0 \\ 0 & \mu_2 & 1 - 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2^* \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{90}$$

This can be satisfied if the constraint set (a) of D_3^{cor} holds (by construction we have $\underline{x} = a_{12}$). The feasible set $\mathcal{S}_{\mathcal{B}_6}$ is then determined by

$$\begin{array}{lrcllcl}
1. & 2a_{22} & \geq & a_{12} & \Longrightarrow & a_{12} & < & \frac{2}{3} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \geq & 2a_{11} \\
3. & \mu_2 & < & \underline{x} & \Longrightarrow & a_{12} & \geq & \frac{1}{2} \\
4. & \mu_1 + 2a_{22} - a_{12} & \geq & \mu_2 & \Longrightarrow & a_{12} & < & \frac{3}{5} + \frac{a_{11}}{5} \\
5. & a_{23} & < & a_{13} & \Longrightarrow & a_{12} & < & 1 - a_{11} \\
6. & 1 - \mu_2 + a_{23} & \leq & 2a_{13} & \Longrightarrow & a_{12} & < & \frac{3}{4} - a_{11} \\
7. & \frac{1}{2}(1 - \mu_2 + a_{13}) & > & 1 - \underline{x} & \Longrightarrow & a_{12} & > & \frac{1}{2} + a_{11}
\end{array}$$

After comparison of the inequalities, only 4. and 7. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_6} \quad = \quad \left\{\left[0, \tfrac{1}{8}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + a_{11}\right) \cup \left[\tfrac{1}{8}, \tfrac{1}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{4} - a_{11}\right)\right\} \tag{91}$$

(See figure 10)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_6) > \frac{1}{2} - \epsilon \tag{92}$$

$\mathcal{B}_7$) Suppose $D_2 = (1, 0, 0)$ and $D_3 = (0, \mu_2, 1 - \mu_2)$ is a single proposal equilibrium $\Longrightarrow$ This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ 0 & \mu_2 & 1 - 1 - \mu_2 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 1 & 2^* & 3 \end{pmatrix} \tag{93}$$

This can be satisfied if the constraints of D_3^{μ} hold. The feasible set $\mathcal{S}_{\mathcal{B}_7}$ is then determined by

$$\begin{array}{lrcllcl}
1. & \mu_2 & < & \mu_1 & \Longrightarrow & a_{12} & < & 1 + a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & < & \frac{1}{2} \\
3. & a_{23} & \leq & a_{13} & \Longrightarrow & a_{12} & \leq & 1 - a_{11} \\
4. & 1 - \mu_2 + a_{23} & > & 2 * a_{13} & \Longrightarrow & a_{12} & > & \frac{2}{3} - \frac{4}{3}a_{11} \\
5. & \mu_2 & > & \frac{3}{8} & \Longrightarrow & a_{12} & > & \frac{3}{4}
\end{array}$$

After comparison of the inequalities only 5. is remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{B}_7} \quad = \quad \left\{\left[0, \tfrac{1}{4}\right] \times \left(\tfrac{3}{4}, 1 - a_{11}\right]\right\} \tag{94}$$

(See figure 11)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{B}_7) > \frac{3}{8} \tag{95}$$

Together we have shown that $\forall\ D_1 \in \mathcal{A} \cup \mathcal{B}\ \exists$ a proposal D_2^s of player 2, such that ρ is symmetric and $\pi_2 > \frac{3}{8}$.

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7.2.2 Proof of Corollary 3

1. Suppose $D_1 \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5$ and D_2 such that ρ is non-symmetric and the best reaction of player 2, then $\pi_2 \geq \frac{1}{2}$, because with D_2 such that ρ is symmetric player 2 obtains at least $\pi_2 = \frac{1}{2}$ (see proof of proposition 3). Furthermore we have from the proof of proposition 2 $\pi_3 > \frac{1}{4}$. Together with the resource constraint we

   obtain $1 = \pi_1 + \pi_2 + \pi_3 > \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \ \lightning$

2. Suppose $(D_1 \in \mathcal{A} \cup \mathcal{B}) \backslash D_1 \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5$ and D_2 such that ρ is non-symmetric and $\pi_1 \geq \frac{1}{4}$.

   At first note that $\pi_1 \geq \frac{1}{4}$ is only possible if the constraints for D_3^o or $D_3^{L^\epsilon}$ hold, because otherwise from prop (..) we have $\pi_1 = 0$ and $\pi_2 \geq \frac{3}{8}$ since the minimum payoff with D_2 such that ρ is symmetric is $\pi_2 = \frac{3}{8}$.

   (a) D_2 such that the constraints D_3^o hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \Longrightarrow a_{21} + a_{22} \leq \underline{x}$

   and $a_{21} = \pi_1 \geq \frac{1}{4}$, $a_{22} = \pi_2 \geq \frac{3}{8} \Longrightarrow \frac{5}{8} \leq \underline{x} \leq \frac{1}{2} \ \lightning$

   (b) Suppose D_2 such that the constraints of $D_3^{L^\epsilon}$ hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $D_3 = (0, 2a_{12} - a_{22} + \epsilon, 1 - (2a_{12} - a_{22} + \epsilon))$ is the best reaction of player 3 and $\pi_1 = \frac{1}{3}(a_{11} + a_{21})$ and $\pi_2 = \frac{1}{3}(2a_{12} - a_{22} + \epsilon + a_{12} + a_{22}) = a_{12} + \frac{\epsilon}{3}$.

      i. Suppose $D_2 \in\in \mathcal{A} \backslash (\mathcal{A}_1 \cup \mathcal{A}_2) \Longrightarrow a_{12} \leq \frac{1}{6} \Longrightarrow$

      $\pi_2 = \frac{1}{6} + \frac{\epsilon}{3} < \frac{3}{8}$ $(\epsilon < \frac{5}{8})$ $\lightning$

      ii. Suppose $D_2 \in\in \mathcal{B} \backslash (\mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5) \Longrightarrow a_{11} \leq \frac{3}{8}$ and $2a_{21} < a_{11}$ (constraints

      of $D_3^{L^\epsilon}$) $\Longrightarrow \pi_1 < \frac{1}{3}(\frac{3}{8} + \frac{3}{16}) \leq \frac{3}{16} \ \lightning$

   (c) Suppose D_2 such that the constraints of $D_3^{L^\epsilon}$ hold $\Longrightarrow a = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $D_3 = (2a_{11} - a_{21} + \epsilon, 1 - (2a_{11} - a_{21} + \epsilon))$ and $\pi_2 = \frac{1}{3}(a_{12} + a_{22})$

      i. Suppose $D_2 \in\in \mathcal{A} \backslash (\mathcal{A}_1 \cup \mathcal{A}_2) \Longrightarrow a_{12} \leq \frac{1}{6}$ and $\pi_2 \leq \frac{1}{9}$

      since $a_{22} \leq a_{12}$. $\lightning$

      ii. Suppose $D_2 \in\in \mathcal{B} \backslash (\mathcal{B}_1 \cup \mathcal{B}_3 \cup \mathcal{B}_5) \Longrightarrow a_{12} \leq \frac{3}{4}$ and $\pi_2 < \frac{3}{8}$ since $2a_{22} < a_{12}$.

■

7.2.3 Proof of Proposition 4 and Corollary 4

Suppose $D_1 \in \mathcal{C}$

$\mathcal{C}_1$) $a_{11}+a_{12} \le \frac{1}{2}$ and $D_2 = (1-a_{22}, a_{22}, 0)$ with $1-(2a_{22}-a_{12}) = \frac{1}{2}(1-a_{11}-a_{12}) \Longrightarrow a_{22} = \frac{1}{4}(1+3a_{12}+a_{11}-2\epsilon)$ and $D_3 = (0, 2a_{22}-a_{12}+\epsilon', 1-(2a_{22}-a_{12}+\epsilon'))$ is a single proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1-a_{22} & a_{22} & 0 \\ 0 & 2a_{22}-a_{12}+\epsilon' & 1-(2a_{22}-a_{12}+\epsilon') \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{96}$$

This can be satisfied if the constraint set (i) of D_3^3 hold with $a_{22} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_1}$ is then determined by

$$\begin{array}{lrcl@{\quad}c@{\quad}rcl}
\mathbf{1}. & a_{11}+a_{12} & \le & a_{22} & \Longrightarrow & a_{12} & < & 1-3a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \le & 1-3a_{11} \\
\mathbf{3}. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{12} & < & \frac{1}{5}+\frac{1}{5}a_{11} \\
4. & 2a_{22}-a_{12} & < & 1 & \Longrightarrow & a_{12} & \le & 1-a_{11} \\
5. & 1-(2a_{22}-a_{12}+\epsilon') & > & \frac{1}{2}(a_{13}+a_{23}) & \Longrightarrow & \epsilon' & < & \epsilon \\
6. & a_{22} & \le & a_{21} & \Longrightarrow & a_{12} & \le & \frac{1}{3}-\frac{1}{3}a_{11} \\
7. & 2a_{21}-a_{11} & > & 2a_{22}-a_{12} & \Longrightarrow & a_{12} & \le & \frac{1}{2}-a_{11}
\end{array}$$

After comparison of the inequalities, only 1. and 3. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_1} \quad = \quad \left\{\left(0,\tfrac{1}{4}\right] \times \left[0, \tfrac{1}{5}+\tfrac{1}{5}a_{11}\right) \cup \left[\tfrac{1}{4},\tfrac{1}{3}\right) \times \left[0, 1-3a_{11}\right)\right\} \tag{97}$$

(See figure 13)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{C}_1) > \frac{1}{2} \tag{98}$$

$\mathcal{C}_2$) $D_2 = (1-a_{22}, a_{22}, 0)$ with $2a_{21}-a_{11} = 1 \Longrightarrow a_{21} = \frac{1}{2}(1+a_{11})$ and $a_{22} = \frac{1}{2}(1-a_{11})$. $D_3 = (0, 2a_{22}-a_{12}+\epsilon, 1-(2a_{22}-a_{12}+\epsilon))$ is a single proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1-a_{22} & a_{22} & 0 \\ 0 & 2a_{22}-a_{12}+\epsilon & 1-(2a_{22}-a_{12}+\epsilon) \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{99}$$

This can be satisfied if the constraint set (ii) of D_3^3 holds with $a_{22} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_2}$ is then determined by

$$\begin{array}{lrcllcl}
\mathbf{1}. & a_{11}+a_{12} & \leq & a_{22} & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-\frac{3}{2}a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{11} & < & \frac{1}{3} \\
\mathbf{3}. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} & \geq & \frac{1}{4}-\frac{1}{4}a_{11} \\
4. & 2a_{22}-a_{12} & < & 1 & \Longrightarrow & a_{12} & > & -a_{11} \\
5. & 2a_{21}-a_{11} & \geq & 1 & \Longrightarrow & 1 & \geq & 1 \\
\mathbf{6}. & 1-(2a_{22}a_{12}+\epsilon) & \geq & \frac{1}{2}(a_{13}+a_{23}) & \Longrightarrow & a_{12} & > & \frac{1}{3}-a_{11} \\
7. & a_{22} & \leq & a_{21} & \Longrightarrow & a_{11} & \geq & 0 \\
8. & 2a_{22}-a_{12} & \geq & \frac{1}{2} & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-a_{11}
\end{array}$$

After comparison of the inequalities, only 1., 3. and 6. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_2} \quad = \quad \left\{\left[0,\tfrac{1}{9}\right]\times\left(\tfrac{1}{3}-a_{11},\tfrac{1}{2}-\tfrac{3}{2}a_{11}\right]\cup\left(\tfrac{1}{9},\tfrac{1}{5}\right]\times\left[\tfrac{1}{4}-\tfrac{1}{4}a_{11},\tfrac{1}{2}-\tfrac{3}{2}a_{11}\right]\right\} \tag{100}$$

(See figure 14)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{C}_2) > \frac{1}{2} \tag{101}$$

$\mathcal{C}_3$) $D_2=(1-a_{22},a_{22},0)$ with $2a_{22}-a_{12}=\frac{1}{2}\Longrightarrow a_{21}=\frac{1}{4}(3-2a_{11})$ and $a_{22}=\frac{1}{4}(1+2a_{11})$. $D_3=(0,2a_{22}-a_{12}+\epsilon,1-(2a_{22}-a_{12}+\epsilon))$ is a single proposal equilibrium. This requires the following rank matrix:

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1-a_{22} & a_{22} & 0 \\ 0 & 2a_{22}-a_{12}+\epsilon & 1-(2a_{22}-a_{12}+\epsilon) \end{pmatrix} \quad\longrightarrow\quad \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2^* \end{pmatrix} \tag{102}$$

This can be satisfied if the constraint set (ii) of D_3^3 holds with $a_{22}=\underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_3}$ is then determined by

$$\begin{array}{lrcllcl}
1. & a_{11}+a_{12} & \leq & a_{22} & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-2a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & < & \frac{3}{2}-4a_{11} \\
3. & 2a_{12} & \geq & a_{22} & \Longrightarrow & a_{12} & \geq & \frac{1}{6} \\
4. & 2a_{22}-a_{12} & < & 1 & \Longrightarrow & \frac{1}{2} & < & 1 \\
5. & 2a_{21}-a_{11} & \geq & 1 & \Longrightarrow & a_{12} & \leq & \frac{1}{2}-a_{11} \\
6. & 1-(2a_{22}a_{12}+\epsilon) & \geq & \frac{1}{2}(a_{13}+a_{23}) & \Longrightarrow & a_{12} & > & -a_{11} \\
7. & a_{22} & \leq & a_{21} & \Longrightarrow & a_{12} & \geq & \frac{1}{2}
\end{array}$$

After comparison of the inequalities, only 1. and 3. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_3} \quad = \quad \left\{\left[0,\tfrac{1}{6}\right]\times\left[\tfrac{1}{6},\tfrac{1}{2}-2a_{11}\right]\right\} \tag{103}$$

(See figure 15)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{C}_3) > \frac{1}{2} \tag{104}$$

$\mathcal{C}_4$) $D_2 = (1 - a_{22}, a_{22}, 0)$ with $2a_{22} - a_{12} + \epsilon = 2a_{21} - a_{11} \Longrightarrow a_{21} = \frac{1}{4}(2 + a_{11 - a_{21} + \epsilon})$ and $a_{21} = \frac{1}{4}(2 - a_{11 + a_{21} - \epsilon})$. $D_3 = (0, 2a_{22} - a_{12} + \epsilon', 1 - (2a_{22} - a_{12} + \epsilon'))$ is a single proposal equilibrium. This requires the following rank matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 - a_{22} & a_{22} & 0 \\ 0 & 2a_{22} - a_{12} + \epsilon' & 1 - (2a_{22} - a_{12} + \epsilon') \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \tag{105}$$

This can be satisfied if the constraint set (i) of $D_3^{T_\epsilon}$ holds with $a_{21} = \underline{x}$. The feasible set $\mathcal{S}_{\mathcal{C}_4}$ is then determined by

$$\begin{array}{lrcllcl}
1. & a_{11} + a_{12} & \leq & a_{21} & \Longrightarrow & a_{12} & \leq & \frac{2}{5} - \frac{5}{5}a_{11} \\
2. & 2a_{11} & < & a_{21} & \Longrightarrow & a_{12} & \leq & 2 - 7a_{11} \\
3. & 2a_{12} & < & a_{22} & \Longrightarrow & a_{12} & < & \frac{2}{7} - \frac{1}{7}a_{11} \\
4. & 2a_{22} - a_{12} & < & 1 & \Longrightarrow & a_{12} & \geq & -a_{11} \\
5. & 1 - (2a_{22}a_{12} + \epsilon') & < & \frac{1}{2}(a_{13} + a_{23}) & \Longrightarrow & a_{12} & < & \frac{1}{2} - a_{11} \\
7. & a_{21} & \leq & a_{22} & \Longrightarrow & a_{12} & > & a_{11} \\
8. & 2a_{21} - a_{11} & \leq & 2a_{22} - a_{12} & \Longrightarrow & \epsilon & > & 0
\end{array}$$

After comparison of the inequalities, only 1. and 3. are remaining.

$$\Longrightarrow \quad \mathcal{S}_{\mathcal{C}_4} \quad = \quad \left\{ \left[0, \tfrac{1}{4}\right] \times \left(a_{11}, \tfrac{2}{7} - \tfrac{1}{7}a_{11}\right) \right\} \tag{106}$$

(See figure 16)

Minimum payoff player 2:

$$\pi_2^{min}(\mathcal{C}_4) > \frac{1}{2} \tag{107}$$

$\mathcal{C}_5$)

In the following we show that

Given $(a_{11}, a_{12}) \in \mathcal{S}_{\mathcal{C}_5} = \left\{ \mathcal{C} \setminus \bigcup_{k=1}^{4} \mathcal{S}_{\mathcal{C}_k} \right\}$ the share of player 1 is $\pi_1 = 0$ or ρ is symmetric and $\pi_2 > \frac{3}{8}$.

In order to proof this, we divide $\mathcal{S}_{\mathcal{C}_5}$ into three subsets

$$\begin{array}{l}
\mathcal{S}_{\mathcal{C}_5}^1 = \left\{ (a_{11}, a_{12}) \,, \,|\,, \, a_{11} + a_{12} > \frac{1}{2} \,, \, a_{11} < \frac{1}{2} \,, \, a_{12} < \frac{1}{2} \right\} \\
\mathcal{S}_{\mathcal{C}_5}^2 = \left\{ (a_{11}, a_{12}) \,, \,|\,, \, a_{11} + a_{12} \leq \frac{1}{2} \,, \, a_{12} \geq \frac{2}{7} - \frac{1}{7}a_{11} \,, \, a_{12} > \frac{1}{2} - \frac{3}{2}a_{11} \right\} \\
\mathcal{S}_{\mathcal{C}_5}^3 = \left\{ (a_{11}, a_{12}) \,, \,|\,, \, a_{11} + a_{12} \leq \frac{1}{2} \,, \, a_{12} \geq 1 - 3a_{11} \,, \, a_{11} \geq 0 \right\}
\end{array} \tag{108}$$

(a) Given $(a_{11}, a_{12}) \in \mathcal{S}_{\mathcal{C}_5}^1 \Longrightarrow$ with $D_2 = (\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon, 0)$ the inequalities of $D_3^{x^\epsilon}$ hold and $D_3 = (0, \frac{1}{2} - \epsilon + \epsilon', \frac{1}{2} + \epsilon - \epsilon')$ is the best proposal of player 3 with D_3 being a proposal equilibrium including $\pi_1 = 0$ and $\pi_2 = \frac{1}{2} - \epsilon + \epsilon'$.

i. Suppose player 2 construct his proposal D_2 such that ρ is non-symmetric and $\pi_1 > 0$, then the inequalities of $D_3^{L^\epsilon}$ or D_3^o must hold ($a_{11} = y_1$, $a_{12} = y_2$). But $y_1 + y_2 \leq \underline{x}$ cannot hold since $\underline{x} \leq \frac{1}{2}$.

ii. Suppose player 2 construct his proposal D_2 such that ρ is symmetric and $\pi_2 \leq \frac{3}{8}$, then this cannot be his best proposal, since proposing $D_2 = (\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon, 0)$ with non-symmetric ρ and $\epsilon < \frac{1}{8}$ ensures him a share of $\pi_2 > \frac{3}{8}$.

(b) Given $(a_{11}, a_{12}) \in \mathcal{S}^2_{\mathcal{C}_5} \cup \mathcal{S}^3_{\mathcal{C}_5} \Longrightarrow$ with $D_2 = (1 - (a_{11} + a_{12} - \epsilon), a_{11} + a_{12} - \epsilon, 0)$ the inequalities of $D_3^{\underline{x}^\epsilon}$ hold and $D_3 = (0, a_{11} + a_{12} - \epsilon + \epsilon', 1 - (a_{11} + a_{12} - \epsilon + \epsilon')$ is the best proposal of player 3 with D_3 being a proposal equilibrium including $\pi_1 = 0$ and $\pi_2 = a_{12} + a_{11} - \epsilon + \epsilon'$.

i. player 2 construct his proposal D_2 such that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{T^o}$ hold, with $\pi_2 = a_{12}$. Then this cannot be the best reaction of player 2, since $D_2 = (1 - (a_{11} + a_{12} - \epsilon), a_{11} + a_{12} - \epsilon, 0)$ with $\epsilon < a_{11}$) ensures him a share of $\pi_2 = a_{11} + a_{12} - \epsilon + \epsilon' > a_{12}$

ii. Suppose player 2 construct his proposal D_2 such that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{L^\epsilon}$ hold with $D_3 = (2a_{21} - a_{11} + \epsilon, 0, 1 - (2a_{21} - a_{22} + \epsilon))$ is the best proposal player 3 ($a_{11} = y_1$, $a_{12} = y_2$) $\Longrightarrow$ This cannot be the best reaction of player 2, because his share $\pi_2 = \frac{1}{3}(x_2 + y_2)$ must exceed his share $\pi_2 = y_1 + y_2 + \epsilon - \epsilon'$ generated by $D_2 = (0, \frac{1}{2} - \epsilon + \epsilon', \frac{1}{2} + \epsilon - \epsilon')$. But

$$\frac{1}{3}(x_2 + y_2) \quad \geq \quad y_1 + y_2 \quad \overset{y_2 \geq 1 - 3y_1 \geq \frac{1}{2} - \frac{3}{2}y_1}{\Longrightarrow} \quad x_2 = 1 \overset{x_1 \leq 1 - x_2}{\Longrightarrow} x_1 = 0 = \underline{x} \quad \lightning \tag{109}$$

since $a_{12} \geq \frac{1}{4}$ and $a_{11} + a_{12} \leq \underline{x}$

iii. Suppose player 2 construct his proposal D_2 such that ρ is symmetric and $\pi_2 \leq \frac{3}{8} \Longrightarrow$

α. $a_{11} + a_{12} \leq \frac{3}{8}$

β. $D_3 \in \{D_3^{cor}, D_3^L\}$

Otherwise player 2 can obtain a share of $\pi_2 = a_{11} + a_{12} - \epsilon + \epsilon' > \frac{3}{8} \geq a_{11} > a_{12}$ by choosing $\epsilon < a_{11} + a_{12} - \frac{3}{8}$ and proposing $D_2 = (1 - (a_{11} + a_{12} - \epsilon), a_{11} + a_{12} - \epsilon, 0)$ with non-symmetric ρ. This implies also $D_3 \in \{D_3^{cor}, D_3^L\}$ following from corollary 5, because if $(a_{11}, a_{12}) \in \{(\mathcal{S}^2_{\mathcal{C}_5} \cup \mathcal{S}^3_{\mathcal{C}_5}) \cap [0, \frac{3}{8}] \times [0, \frac{3}{8} - a_{11}]\} = \{[\frac{5}{16}, \frac{1}{3}] \times [1 - 3a_{11}, \frac{3}{8} - a_{11}] \cup [\frac{1}{3}, \frac{3}{8}] \times [0, \frac{3}{8} - a_{11}]\}$ and given symmetric ρ, we have $\underline{x} \leq a_{11} \leq \frac{3}{8} \wedge \underline{x} \leq a_{12} \leq \frac{1}{16}$.

α. Suppose player 2 construct his proposal D_2 such that ρ is symmetric and $D_3 = D_3^{cor}$ is the best proposal of player 3 leading to a correlated equilibrium $C_{23} \Longrightarrow$

This cannot be the best reaction of player 2 because his share $\pi_2(D_3^{cor}) = \frac{1}{2}(\mu_2 + x_2) = \frac{1}{2}(\frac{1}{2}(a_{22} + a_{12}) + a_{22}) = \frac{3}{4}a_{22} + \frac{1}{4}a_{12}$ must exceed $\pi_2(D_3^{\underline{x}^\epsilon}) = a_{11} + a_{12} - \epsilon + \epsilon'$, which implies

$$\begin{array}{rcll} \tfrac{3}{4}a_{22} + \tfrac{1}{4}a_{12} & \geq & a_{11} + a_{12} & \Longrightarrow \quad a_{22} \geq \tfrac{4}{3}a_{11} + a_{12} \\ & \overset{\substack{a_{11} \geq \frac{5}{16} \\ a_{12} \geq 0}}{\geq} & \frac{5}{12} & \end{array} \tag{110}$$

But this is a contradiction of

$$\hat{R}(D_3^{cor}) = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 2^* \\ 1 & 2^* & 3 \end{pmatrix} \tag{111}$$

which implies

$$\begin{array}{rcccl} a_{23} & \geq & a_{13} & \geq & a_{22} \geq \tfrac{4}{3}a_{11} + a_{12} \\ & \overset{a_{11}+a_{12} \leq \frac{3}{8}}{\geq} & \frac{5}{8} & \Longrightarrow & a_{22} \leq \tfrac{3}{8} \end{array} \tag{112}$$

β. Suppose player 2 construct his proposal D_2 such that ρ is symmetric and $D_3 = D_3^{cor}$ is the best proposal of player 3 leading to a correlated equilibrium $C_{13} \Longrightarrow$
This cannot be the best reaction of player 2, since his share is given by $\pi_2 = \frac{1}{2}(a_{12} + a_{32}) \leq a_{12}$ because $\Phi(a_{32}) \leq 2$.

γ. Suppose player 2 construct his proposal D_2 such that ρ is symmetric and $D_3 = D_3^L$ is the best proposal of player 3 leading to a drawing lots equilibrium $\Longrightarrow$
This cannot be the best reaction of player because if $a_{32} = 2a_{12} - a_{22} + \epsilon$ we have $\pi_2(D_3^L) = \frac{1}{3}(a_{12} + 2a_{12} - a_{32}) = a_{12} + \epsilon < \pi_2(D_3^{x^\epsilon})$ and if $a_{32} = 0$ $\pi_2(D_3^L > \pi_2(D_3^{x^\epsilon})$ would imply

$$\begin{array}{rcll} \tfrac{1}{3}(a_12 + a_{22}) & \geq & a_{11} + a_{12} & \Longrightarrow \quad a_{22} \geq 3a_{11} + 2a_{12} \\ & \overset{\substack{a_{11} \geq 0 \\ a_{12} \geq 1 - 3a_{11} \\ a_{12} \leq \frac{3}{8} - a_{11}}}{\geq} & 1 & \end{array} \tag{113}$$

but then we have $a_{23} = 0$, $a_{13} \geq \frac{5}{8}$ and $a_{33} \leq 1$, which cannot lead to a drawing lots equilibrium with $\Phi(a_{33}) = 3$.

iv. Suppose $(a_{11}, a_{12}) \in \mathcal{S}^2_{\mathcal{C}_5}$ and player 2 construct his proposal D_2 such that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{L^\epsilon}(i)$ hold with $D_3 = (0, 2a_{22} - a_{12} + \epsilon, 1 - (2a_{22} - a_{12} + \epsilon))$ is the best proposal player 3 ($a_{11} = y_1$, $a_{12} = y_2$) $\Longrightarrow$

$$\begin{array}{lcccc} 3y_2 & \overset{2y_2 < x_2}{<} & 2x_2 - y_2 & \overset{2x_2 - y_2 < 2x_1 - y_1}{<} & 2x_1 - y_1 \\ & \overset{x_1 < 1 - x_2}{<} & 2(1 - x_2) - y_1 & \overset{2y_2 < x_2}{<} & 2 - 4y_2 - y_1 \end{array} \Longrightarrow y_2 < \frac{2}{7} - \frac{1}{7}y_1 \quad \lightning \tag{114}$$

since $a_{12} \geq \frac{2}{7} - \frac{1}{7}a_{11}$ must hold in $\mathcal{S}^2_{\mathcal{C}_5}$.

v. Suppose $(a_{11}, a_{12}) \in \mathcal{S}^2_{\mathcal{C}_5}$ player 2 construct his proposal D_2 such that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{L^\epsilon}(ii)$ hold with $D_3 = (0, 2a_{22} - a_{12} + \epsilon, 1 - (2a_{22} - a_{12} + \epsilon))$ is the best proposal player 3 $(a_{11} = y_1\, ,\, a_{12} = y_2) \Longrightarrow$

$$\begin{array}{ccccc} 1 & \leq & 2x_1 - y_1 & \overset{x_1 \leq 1 - x_2}{\leq} & 2 - 2x_2 - y_1 \\ & \overset{x \leq \underline{x}}{\leq} & 2 - 2\underline{x} - y_1 & \overset{y_1 + y_2 \leq \underline{x}}{\leq} & 2 - 3y_1 - 2y_2 \end{array} \Longrightarrow y_2 \leq \frac{1}{2} - \frac{3}{2} y_1 \quad \lightning \tag{115}$$

since $a_{12} \leq \frac{1}{2} - \frac{3}{2} a_{11}$ must hold in $\mathcal{S}^2_{\mathcal{C}_5}$.

vi. Suppose $(a_{11}, a_{12}) \in \mathcal{S}^3_{\mathcal{C}_5}$ and player 2 construct his proposal D_2 such that ρ is non-symmetric, $\pi_1 > 0$ and the inequalities of $D_3^{L^\epsilon}$ hold with $D_3 = (0, 2a_{22} - a_{12} + \epsilon, 1 - (2a_{22} - a_{12} + \epsilon))$ is the best proposal player 3 $(a_{11} = y_1\, ,\, a_{12} = y_2) \Longrightarrow$

$$\begin{array}{ccccc} y_1 + y_2 & \leq & \underline{x} & \overset{x_2 \leq \underline{x}}{<} & x_2 \\ & \overset{x_2 \leq 1 - x_1}{\leq} & 1 - x_1 & \overset{2y1 < x_1}{<} & 1 - 2_y 1 \end{array} \Longrightarrow y_2 < 1 - 3y_1 \quad \lightning \tag{116}$$

since $a_{12} \geq 1 - 3a_{11}$ must hold in $\mathcal{S}^3_{\mathcal{C}_5}$.

Altogether we have shown proposition 4 and corollary 4.

■

7.2.4 Proof Proposition 5

If $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow \pi_2 = \frac{1}{4}$ is already shown in corollary 1.

1. Suppose $a = \begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix} \Longrightarrow D_3 = (\epsilon, 0, 1 - \epsilon)$ is the best reaction of player 3 with D_3 being a single proposal equilibrium. $\Longrightarrow \pi_2 = 0$

   The solution set is given by

$$\mathcal{S}_1 \;=\; \{[0, 0] \times [0, 1]\} \tag{117}$$

2. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{llllll} 1. & a_{22} & < & \frac{1}{2} & & \\ 2. & 1 - \mu_1 & > & 2a_{23} \Longrightarrow & a_{22} & > \frac{1}{2} - \frac{3}{4} a_{21} \end{array} \tag{118}$$

then the constraints of D_3^μ hold and $D_3^\mu = (\mu_1, 0, 1 -)\mu_1)$ is the best reaction of player 3 with D_3^μ being a single proposal equilibrium. $\Longrightarrow \pi_2 = 0$

The solution set is given by

$$\begin{aligned}\mathcal{S}_2 \;=\; \{ \;& \left[0, \tfrac{1}{2}\right] \times \left(\tfrac{1}{2} - \tfrac{3}{4}a_{21}, \tfrac{1}{2}\right) \cup \\ & \left(\tfrac{1}{2}, \tfrac{1}{3}\right] \times \left(\tfrac{1}{2} - \tfrac{3}{4}a_{21}, 1 - a_{21}\right] \cup \\ & \left(\tfrac{1}{3}, 1\right) \times [0, 1 - a_{21}]\big\}\end{aligned} \tag{119}$$

3. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcllcl} 1. & a_{22} & < & \frac{1}{2} & & & \\ 2. & 1 - \mu_1 & \le & 2a_{23} & \Longrightarrow & a_{22} & \le & \frac{1}{2} - \frac{3}{4}a_{21} \\ 3. & \frac{1}{2}(1 - \mu_1 + a_{23}) & \le & 1 - a_{21} & \Longrightarrow & a_{22} & \ge & \frac{1}{2}a_{21} \end{array} \tag{120}$$

then the constraints (ii) of $D_3^{\underline{x}}$ hold and $D_3^{\underline{x}} = (\underline{x}, 0, 1 - \underline{x})$ is the best reaction of player 3 with $D_3^{\underline{x}}$ being a single proposal equilibrium. $\Longrightarrow \pi_2 = 0$

The solution set is given by

$$\mathcal{S}_3 \;=\; \left\{\left(0, \tfrac{2}{5}\right] \times \left[\tfrac{1}{2}a_{21}, \tfrac{1}{2} - \tfrac{3}{4}a_{21}\right]\right\} \tag{121}$$

4. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcllcl} 1. & 1 - \mu_1 & \le & 2a_{23} & \Longrightarrow & a_{22} & \le & \frac{1}{2} - \frac{3}{4}a_{21} \\ 2. & \frac{1}{2}(1 - \mu_1 + a_{23}) & > & 1 - a_{21} & \Longrightarrow & a_{22} & < & \frac{1}{2}a_{21} \end{array} \tag{122}$$

then the constraints (a) of D_3^{cor} hold and $D_3^{cor} = (\mu_1, 0, 1 - \mu_1$ is the best reaction of player 3 with C_{23} being a correlated equilibrium. $\Longrightarrow \pi_2 = \frac{1}{2}a_{22} < \frac{1}{5}$

The solution set is given by

$$\mathcal{S}_4 \;=\; \left\{\left(0, \tfrac{2}{5}\right] \times \left[0, \tfrac{1}{2}a_{21}\right) \cup \left(\tfrac{2}{5}, \tfrac{1}{3}\right] \times \left[0, \tfrac{1}{2} - \tfrac{3}{4}a_{21}\right)\right\} \tag{123}$$

5. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcllcl} 1. & a_{22} & \ge & \frac{1}{2} & & & \\ 2. & \mu_1 + 2a_{22} - a_{12} & < & a_{21} & \Longrightarrow & a_{22} & < & \frac{1}{2} + \frac{1}{4}a_{21} \\ 3. & 1 - \mu_1 - (2a_{22} - 1) & > & 2a_{23} & \Longrightarrow & a_{21} & > & 0 \end{array} \tag{124}$$

then the constraints (i) of $D_3^{\mu_i}$ hold and $D_3^{\mu_i} = (\mu_1, 2a_{22}-1+\epsilon, 1-(\mu_1+2a_{22}-1+\epsilon))$ is the best reaction of player 3 with $D_3^{\mu_i}$ being a single proposal equilibrium. $\Longrightarrow$ $\pi_2 = 2a_{22} - 1 + \epsilon < \frac{1}{5} - \epsilon' + \epsilon$

The solution set is given by

$$\mathcal{S}_5 \;=\; \left\{\left(0, \tfrac{2}{5}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{4}a_{21}\right) \cup \left(\tfrac{2}{5}, \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, 1 - a_{21}\right]\right\} \tag{125}$$

6. Suppose $a = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$ and

$$\begin{array}{lrcl} 1. & a_{21} & > & 0 \\ 2. & \mu_1 + 2a_{22} - a_{12} & \leq & a_{21} \implies a_{22} < \frac{1}{2} + \frac{1}{4}a_{21} \end{array} \tag{126}$$

then the constraints (iii) of $D_3^{\underline{x}}$ hold and $D_3^{\underline{x}} = (\underline{x}, 0, 1 - \underline{x})$ is the best reaction of player 3 with $D_3^{\underline{x}}$ being a single proposal equilibrium. $\implies \pi_2 = 0$

The solution set is given by

$$\mathcal{S}_6 \quad = \quad \left\{ \left(0, \tfrac{2}{5}\right] \times \left[\tfrac{1}{2} + \tfrac{1}{4}a_{21}, 1 - a_{21}\right] \right\} \tag{127}$$

Altogether we have shown that $D_2 = (1, 0, 0)$ is the best reaction of player 2 given $D_1 = (0, 1, 0)$

■

8 Literature

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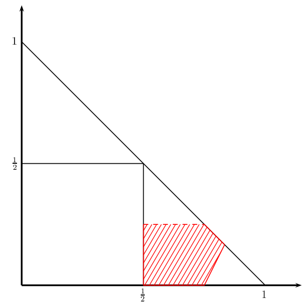

Figure 1: Feasible set $\mathcal{S}_{\mathcal{A}_1}$.

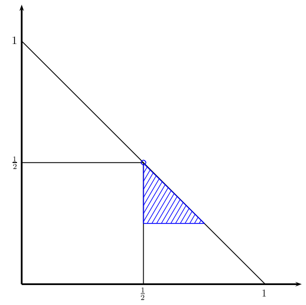

Figure 2: Feasible set $\mathcal{S}_{\mathcal{A}_2}$.

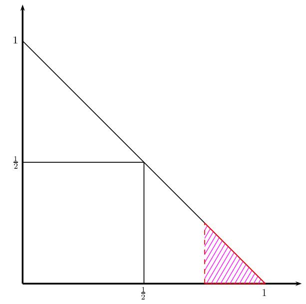

Figure 3: Feasible set $\mathcal{S}_{\mathcal{A}_3}$.

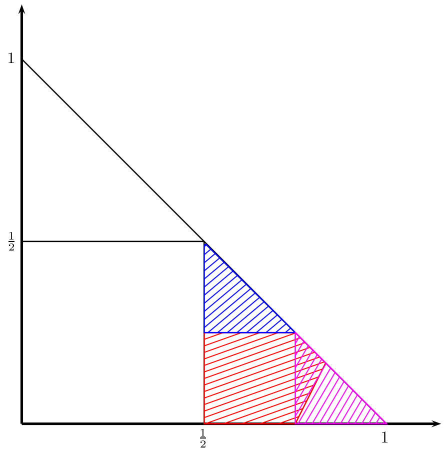

Figure 4: Feasible set $\mathcal{S}_{\mathcal{A}}$.

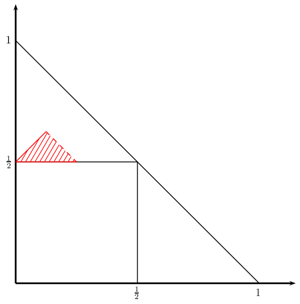

Figure 5: Feasible set $\mathcal{S}_{\mathcal{B}_1}$.

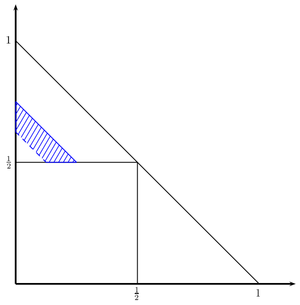

Figure 6: Feasible set $\mathcal{S}_{\mathcal{B}_2}$.

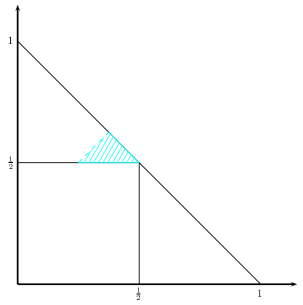

Figure 7: Feasible set $\mathcal{S}_{\mathcal{B}_3}$.

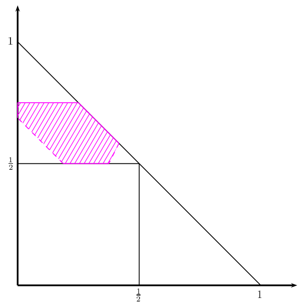

Figure 8: Feasible set $\mathcal{S}_{\mathcal{B}_4}$.

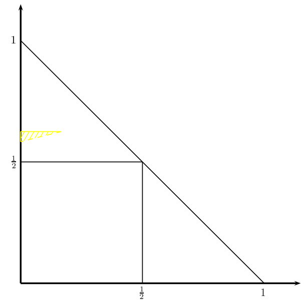

Figure 9: Feasible set $\mathcal{S}_{\mathcal{B}_5}$.

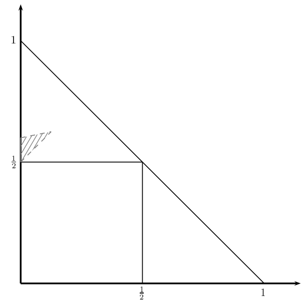

Figure 10: Feasible set $\mathcal{S}_{\mathcal{B}_6}$.

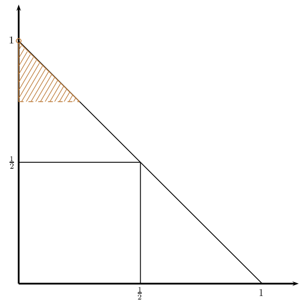

Figure 11: Feasible set $\mathcal{S}_{\mathcal{B}_7}$.

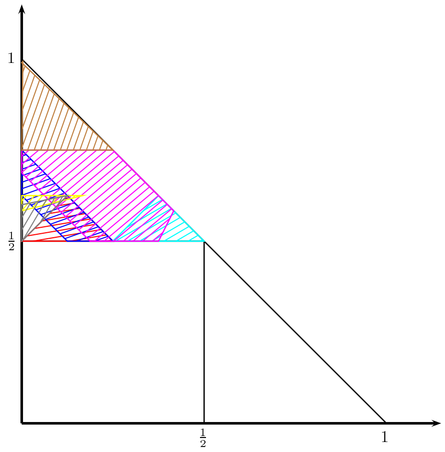

Figure 12: Feasible set $\mathcal{S}_{\mathcal{B}}$.

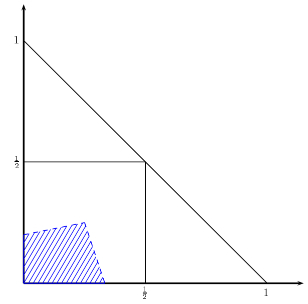

Figure 13: Feasible set $\mathcal{S}_{\mathcal{C}_1}$.

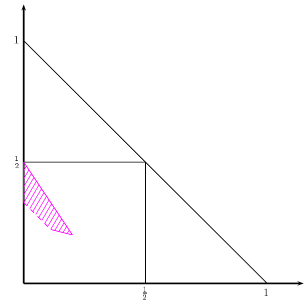

Figure 14: Feasible set $\mathcal{S}_{\mathcal{C}_2}$.

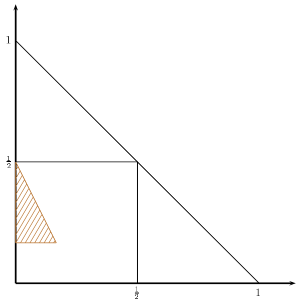

Figure 15: Feasible set $\mathcal{S}_{\mathcal{C}_3}$.

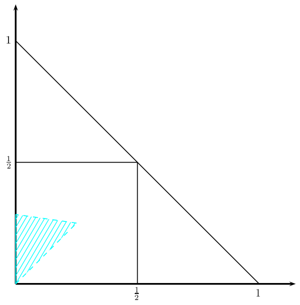

Figure 16: Feasible set $\mathcal{S}_{\mathcal{C}_4}$.